

Introduction

The first thing we want to say is that the words ‘answer’ and ‘solution’ mean two different things. An **answer** is a short reply that gives the answer to the question asked. For example, in question 1a, the answer is two. Sometimes we say a few extra words that roughly give some idea of how you might get the answer.

On the other hand, a **solution** requires more effort. It is where the story of how to get the answer is written out. Sometimes there are only a few lines involved, like in the solution to 1a below, but in some solutions, there are many lines, and you’ll see some of these below too.

At the end of a solution, we may add some “**other thoughts**” that you might have discovered for yourself. These are written in blue text. These may also be things that you hadn’t noticed, and they may be worth thinking about.

The solution methods here begin with trial and error followed by some guessing (conjecturing) about how bigger cases (more steps) can be found from the smaller ones. This conjecturing leads to a general pattern for an arbitrary (any) number of steps. At first our problems support you through this process, though over time you will be ready to do this on your own (See Problem 15 onwards). Drawing diagrams is an important tool that is also used here. And don’t forget to try to find a systematic way to do the problem. The point of being systematic is that you can be sure you have found **all** of the answers and/or have covered all the cases. It also makes sure that you don’t repeat an answer or a case.

Problem 1

Leo the rabbit is climbing up a flight of stairs. Leo can only hop up one or two steps each time he hops. We'll call these 1-hops or 2-hops. By the way, he never hops down, only up.

a. How many different ways can Leo hop up a flight of two steps?

If there are only two steps Leo can 1-hop twice or 2-hop once. So there are **two** different ways that Leo can reach the top of the stairs. You can see how this works from the diagram.

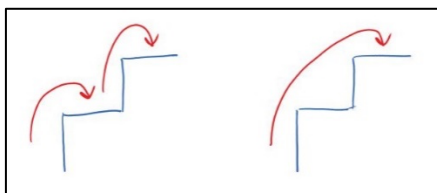


Diagram 1: The 2 moves for 2 steps.

Observation. Notice that the diagram above starts off with a 1-hop, but Leo could have started off with a 2-hop. These are different ways for Leo to achieve his goal. As a result, there are only two ways for Leo to get to the top.

b. How many different ways can Leo hop up a flight of three steps?

Think systematically here too. Just like before, Leo can either do no 2-hops or one 2-hop. The diagram below shows how there is just one way to do no 2-hops, and two ways to do one 2-hop.

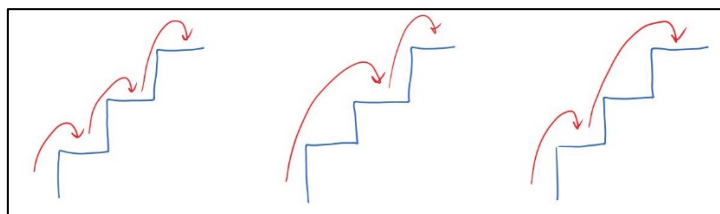


Diagram 2: The 3 moves for 3 steps.

To represent the solution more efficiently we can use a table. The table below shows the number of ways of ordering each situation.

Note that 111 represents 1-hop, 1-hop, 1-hop; 21 represents 2-hop, 1-hop; and 12 shows a 1-hop followed by a 2-hop.

Zero x 2-hops	One x 2-hop
111	21, 12

Observation: We see that writing the numbers in a table is quicker than drawing out the diagrams. For future questions we'll just use the table.

We are confident we haven't missed anything as Leo can't do more than one 2-hop, and there are no other ways of arranging any of the numbers in each column.

We have our answer to the question: Leo can climb the three steps in **three** different ways.

Note. In the second and third steps in Diagram 2, Leo only uses one 1-hop and one 2-hop. However, because the order of these hops is different, Leo decides reasonably that they are different.

Observations.

(i) We have started using even more efficient ways to represent the hops. We have 111 representing 1-hop, 1-hop, 1-hop, and 21 representing 2-hop, 1-hop. This makes the writing even easier.

(ii) In being systematic, Leo has chosen to build up the number of 2-hops from zero to the maximum. Then, when we list the possible options, we always start with the 2-hops at the start and gradually move them up. It's a good idea to be consistent and the next time there is a choice we will build up in this same way.

Interesting idea. Does the 2 ways to get up the stairs for the 2 steps and the 3 ways for the 3 steps suggest that for 4 steps there will be 4 ways?

c. How many different ways can Leo hop up a flight of four steps?

Let's use our systematic approach to produce a new table. Remember, zero 2-hops, then one 2-hop and now we also have two 2-hops.

Zero x 2-hops	One x 2-hop	Two x 2-hops
1111	211, 121, 112	22

Excellent! This table is giving us a quick way to list all options.

There are **five** ways that Leo can climb the four steps.

Observation. $2 + 3 = 5$. Is that an accident? Whatever, it isn't the 4 we might have thought would be the answer after parts a and b.

Interesting idea. From what you have seen so far, what is your guess for the number of ways needed with five steps? What arguments might you give for or against the answer to part d being 6, 7, 8, 9, or 10?

d. How many different ways can Leo use to hop up a flight of five steps?

Time for another table...

Zero x 2-hops	One x 2-hop	Two x 2-hops
11111	2111, 1211, 1121, 1112	221, 212, 122

We see that there are **eight** ways that Leo can climb the five steps.

Interesting ideas.

(i) Notice again how we always keep the 2-hops at the start where possible!!?? So, when there were two 2-hops they were both at the start, then one moved up then they both moved up.

(ii) From what you have seen so far, what is your guess for the number of ways needed with six steps? What arguments might you give for or against the answer to part e being less than 10, 10, 11, 12, 13, 14, more than 14? (It might help to know how many ways Leo can reach the top of a stair with only one step.)

(iii) $3 + 5 = 8$?

e. How many different ways can Leo use to hop up the flight of 6 steps?

Zero x 2-hops	One x 2-hop	Two x 2-hops	Three x 2-hops
111111	21111, 12111, 11211, 11121, 11112	2211, 2121, 2112 1221, 1212 1122	222

We see that there are **thirteen** ways that Leo can climb the six steps.

Interesting ideas.

(i) We had to work quite a bit harder for the two 2-hops! Again, we've been systematic by keeping the 2-hops at the start and gradually moving their numbers up. However, notice how we consider all the cases where the first 2-hop is in the first position and gradually move the second 2-hop up before we consider moving the first 2-hop up.

(ii) $5 + 8 = 13$?

(iii) what about stairs with 10 steps?

f. How many different ways can Leo use to hop up the flight of 10 steps?

Okay, so now there are a lot of cases to consider! We have started to see a pattern though so let's start by making a conjecture (a prediction). We've noticed that:

The number of ways to climb **two** steps + The number of ways to climb **three** steps = The number of ways to climb **four** steps.

And...

The number of ways to climb **three** steps + The number of ways to climb **four** steps = The number of ways to climb **five** steps.

AND...

The number of ways to climb **four** steps + The number of ways to climb **five** steps = The number of ways to climb **six** steps.

So let's continue this pattern and see, if this pattern holds, how many ways there are for Leo to climb the 10 steps. Before, we do this let's come up with a quicker way to write these sums. Instead of "The number of ways to climb **three** steps + The number of ways to climb **four** steps = The number of ways to climb **five** steps."

Let's write, " $W(3)+W(4)=W(5)$ ".

So, we predict that

- $W(7)=W(6)+W(5)=13+8=21$
- $W(8)=W(7)+W(6)=21+13=34$
- $W(9)=W(8)+W(7)=34+21=55$

- $W(10)=W(9)+W(8)=55+34=89$

But how can we be sure this works and will calculate the number of ways that Leo can climb the 10 steps?? There are two ways we can go about this. One, list all the options for the ten steps, or two come up with a **reason** why we can add the previous answers to get the new ones. Option one is time consuming, though somewhat straight forward. Option two requires some deep thinking. Let's do both!

Writing out all, what we believe to be, all 89 possibilities would be quite time consuming, so instead we look at the initial positions and how many different solutions we can make by moving the final two to the right. For example, 1222111 (x4) captures the following solutions: 1222111, 1221211, 1221121, 1221112. Here's what we get working every so systematically!

Zero x 2-hops	One x 2-hop	Two x 2-hops	Three x 2-hops	Four x 2-hops	Five x 2-hops
111111	211111111 (x9) (moving the 2 to the right)	22111111 (x7) 12211111 (x6) 11221111 (x5) 11122111 (x4) 11112211 (x3) 11111221 (x2) 11111122 (x1) (moving the second 2 right in each case)	2221111 (x5) 2122111 (x4) 2112211 (x3) 2111221 (x2) 2111122 (x1) 1222111 (x4) 1212211 (x3) 1211221 (x2) 1211122 (x1) 1122211 (x3) 1121221 (x2) 1121122 (x1) 1112221 (x2) 1112122 (x1) 11112222 (x1) (moving the third 2 to the right each time)	222211 (x5) 222121 (x4) 221221 (x3) 212221 (x2) 122221 (x1) (moving the first 1 to the left)	22222

Here we have **89** ways! Brilliant!!

Now we move onto our second option, explaining why we can add the two previous answers to get the new one. A specific case of this is $W(3)+W(4)=W(5)$, though in general we are saying

$$W(n-1)+W(n-2)=W(n)$$

The pictures below show the solution for five steps.

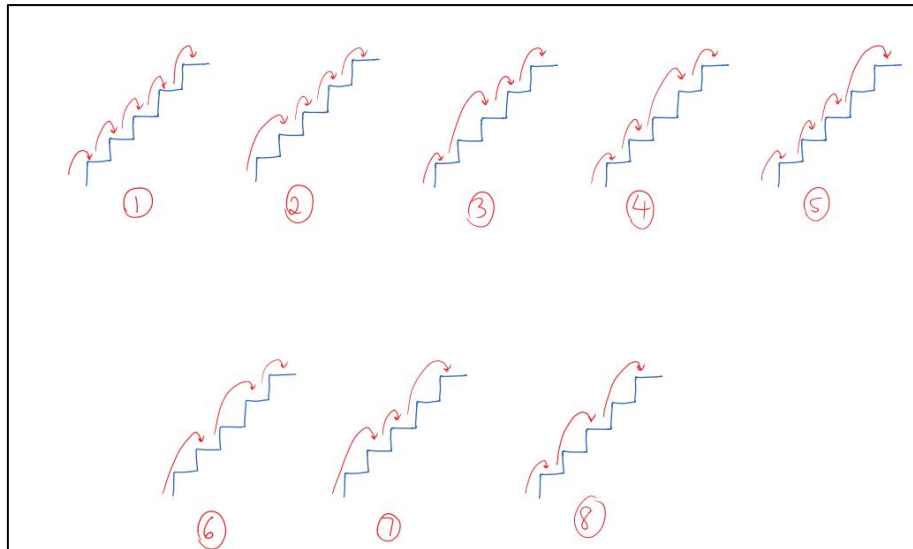


Diagram 3 The 8 moves for 5 steps.

It is possible to separate this solution into two cases. Case 1: Leo starts with a 1-hop, and Case 2: Leo starts with a 2-hop.

In Case 1, Leo is left with 4 steps to climb, and from our previous work we know that there are $W(4)=5$ ways

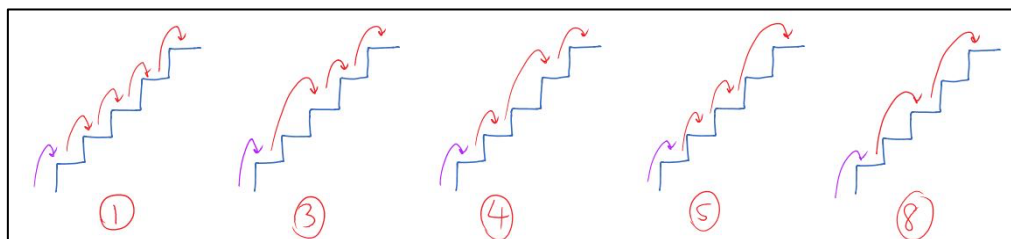


Diagram 4. The 5 steps, beginning with a 1-hop.

In Case 2, Leo is left with three steps to climb, and from our previous work we know that there are $W(3)=3$ ways of climbing these three steps.

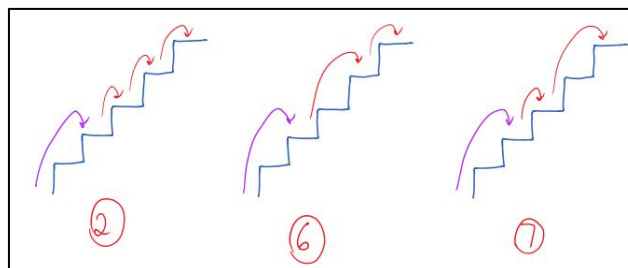


Diagram 5. The 3 steps, beginning with a 2-hop.

In the same way, we can build the solution for 10 steps from $W(8)$ and $W(9)$. Case 1: Leo starts with a 2-hop and then has $W(8)$ ways to complete the remaining 8 steps and Case 2: Leo starts with a 1-hop and then has $W(9)$ ways to complete the remaining 9 steps. So $W(9)+W(8)=W(10)$.

If you think that this is interesting, look up Fibonacci numbers on the web.

Further Exploration: What if Leo could hop 1, 2 or 3 steps at a time?

How happy we are that we decided to work a bit harder and find a formula for the number of ways Leo can hop up n steps using 1-hops and 2-hops! We predict that to calculate the number of ways Leo can climb n steps is $W(n)=W(n-1)+W(n-2)+W(n-3)$. The reason for this is that Leo's climb can be broken into three different cases. Case 1, Leo starts with a 1-hop and has $W(n-1)$ ways to climb the remaining $n-1$ steps. Case 2, Leo starts with a 2-hop and has $W(n-2)$ ways to climb the remaining $n-2$ steps. Case 3, Leo starts with a 3-hop and has $W(n-3)$ ways to climb the remaining $n-3$ steps.

While we are very confident that the formula works, it doesn't hurt to work double check our work by individually listing the ways Leo hops up these steps using 1, 2 or 3 hops at a time.

Number of steps	Only 1-hops	Only 1 and 2-hops	1, 2 and 3-hops	Number of ways
1	1			$W(1)=1$
2	11	2		$W(2)=2$
3	111	21, 12	3	$W(3)=4$
4	1111	211, 121, 112 22	31, 13	$W(4)=7$ $=W(3)+W(2)+W(1)$
5	11111	2111, 1211, 1121, 1112 221, 212, 112	311, 131, 113 32, 23	$W(5)=13$ $=W(4)+W(3)+W(2)$

Note: Finding the solution by adding previous cases is also systematic. However, I wonder if we would have seen this new (and more efficient solution process) if we hadn't been able to work systematically to answer the simpler problems and see the pattern in the answers? It is quite common to improve methods for solving problems as you get further into them.

Problem 2

Imagine a long thin strip of paper stretched out in front of you, left to right. Imagine folding this piece of paper so that the left end meets the right. Now press the strip flat so that it is folded in half with a crease. Continue to fold the folded piece of paper in half two more times.

- a. How many creases are there after one fold?

You can imagine this, of course, but after the first couple of folds it's hard to keep track of what is going on. From Diagram 1, and your imagination, you should see only one crease.

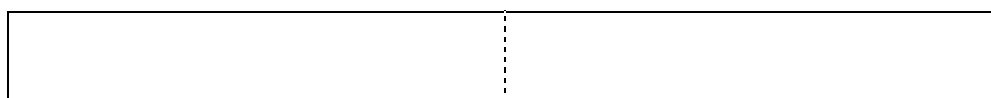


Diagram 1. One fold produces one crease.

- b. How many creases are there after 2 folds?

Again, the solution is to fold the strip twice and then simply count the number of folds. From Diagram 2 you can see that there are three creases at this stage.



Diagram 2. Two folds produces three creases.

- c. How many creases are there after 3 folds?

Repeating the folding method of the solution, this time seven creases have been produced.

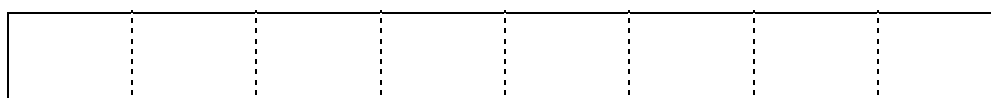


Diagram 3. Three folds produce seven creases.

- d. How many creases are there after 10 folds?

Just when you thought this was all very easy, things have suddenly got hard. It's all well and good to say make ten folds, but even if you have a very, very long piece of paper, it seems that folding it more than 7 times is close to impossible. So let's forget your actual paper and give the question a moment's thought. In fact, go right back to the start. What has been going on? Where are all of these creases coming from?

It's easy to see where the first crease comes from. It's sitting there in the middle of Diagram 1 because the strip was just folded in half.

In the next fold, the doubled strip is folded, and creases magically appear on each part of the doubled strip. Since $1 + 2 = 3$, you now have three creases. But you also now have four strippy bits flapping around. Folding produces a new crease on each of the four flappy bits. So now you have $1 + 2 + 4 = 7$ creases. Now you can go on blithely writing down doubles until you get

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512.$$

And you can just add that little lot up to get 1023 creases.

Observations.

(i) Why do the numbers keep doubling as you go to complete the ten folds?

(ii) Why did we stop at 512?

(iii) Wouldn't you like to have a nicer way to get the sum?

(iv) $1 + 2 = 3 = 4 - 1$; $1 + 2 + 4 = 7 = 8 - 1$; $1 + 2 + 4 + 8 = 15 = 16 - 1$. Does this go on forever? If so, why?

e. How many creases are there after n folds?

Perhaps the answers that lead to the number of creases in ten folds might help you find a general rule.

Folds	1	2	3	4	5	6	7	8	9	10
Creases	1	3	7	15	31	63	127	255	511	1023

Each of these values is of the form $2^n - 1$ where n is the number of folds. This is exciting because if it is true we have a way of working out how many creases for any number of folds. Before we can use this formula with confidence we need to work out if it holds for all cases, so let us dig a little deeper.

One way to do this is to consider why the number of additional creases always is twice the last number of folds. The first fold gives you the one crease on the end and two flappy bits. The next fold folds each flappy bit to give you a crease on the previous two flappy bits and doubles the number of flappy bits. This happens every time we fold. We get an extra power of 2 number of creases from the previous flappy bits and it doubles the number of flappy bits.

This should convince you that the number of creases after 10 folds is indeed, C for creases,

$$C = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512.$$

To stop us having to add all those numbers up, note that

$$2C = 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024.$$

And here is a nice trick. Subtract C from $2C$.

$$\begin{aligned} 2C - C &= (2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024) \\ &\quad - (1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512). \end{aligned}$$

For goodness' sake don't add up all the numbers in the two brackets. Most of them cancel out. In fact

$$2C - C = C = 1024 - 1 = 1023.$$

Do you see how doing a little bit extra can save you a lot of time? Mathematicians are always trying to find neat ways to make life simpler for them. That's because they are lazy or maybe just clever.

But the question we're asked to solve is how many creases we will make with n folds. There are two things to ask now. First what are these numbers 1, 2, 4, 8, and so on? Is there anything they have in common? Second where do they stop?

Now the numbers 1, 2, 4, 8, and so on, get doubled every time you make a fold. So, they are all powers of 2. Consequently, $1 = 2^0$, $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, and so on.

As for stopping, after one fold the stopping number is 2^0 . (And that's also the starting number!)

- After two folds the stopping number is 2^1 .
- After three folds the stopping number is 2^2 .
- After four folds the stopping number is 2^3 .
- After five folds the stopping number is 2^4 .

Do you get the idea? The power of the last number is one less than the number of folds. So with n folds, the last number is 2^{n-1} .

For n folds we have

$$C = 2^0 + 2^1 + 2^2 + \dots + 2^{n-3} + 2^{n-2} + 2^{n-1}.$$

Now do the clever $2C - C$ trick to give $C = 2^n - 1$.

Just to make sure that that's OK, check C for $n = 1, 2, 3$ and 10.

Note. How many folds can you actually make in practice? What's the best number of folds anyone in the class can do? Three, four, ...? Can anyone actually **do** 10 folds? Is there a world record?

Further Exploration: How many creases would there be if you alternated between horizontal and vertical folds?

We show the first few of these in the Diagram 4.

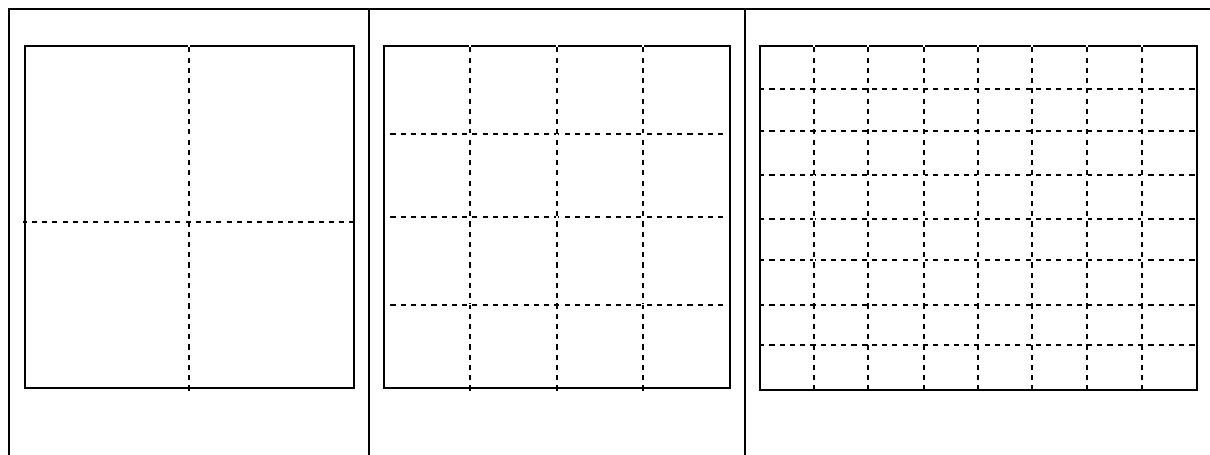


Diagram 4. Folding both ways for $n = 1, 2$ and 3 times.

We first notice that the number of vertical folds is identical to the number of horizontal folds in each case. So, let's just find the number of vertical folds and double it. So, how many vertical folds are there? Well, let's count how many vertical folds there are in the top row. We have 1, then 3, then 7.... Ah it's our original problem. This is great! We now know that there are $2^n - 1$ folds in the top row. How many rows are there? There are 2, then 4, then 8. Excellent these are just doubling each time, so we have 2^n rows. Multiplying the number of vertical creases in each row by the number of rows gives us the total number of vertical creases: $(2^n - 1) \times 2^n$. As said earlier, the number of horizontal

creases is the same as the number of vertical creases, so we can just double this to get the total number of creases. So the general answer, for n folds, turns out to be $(2^n - 1) \times 2$.

Check this formula out for $n = 1, 2, 3, 4, 5$ and 6 .

Problem 3

In Diagram 1, there are three lily pads. The one in the middle is empty. On the left side there is a green frog and on the right side there is a blue frog.

For some reason they want to change places. To do this can they either jump to an empty lily pad or jump over the other frog onto an empty square on the other side.

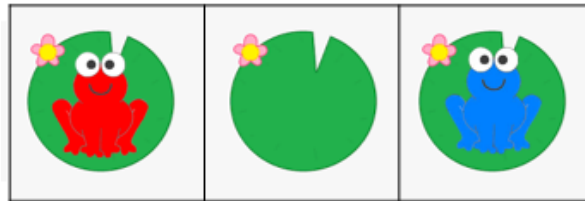


Diagram 1. Two frogs on three lily pads.

- a. What is the **smallest** number of jumps do they have to make to be able to swap places?

Note to teacher. Now it may be easy to imagine the situation and the jumps in this first case. But life is going to get harder, so get the students to think how they might go about this problem and what things would help them to do it. If at any time anyone gets stuck, encourage them to (i) draw the pictures of the frogs as they move them; (ii) use manipulatives (blocks are useful, but so are pieces of torn paper); (iii) use the children as frogs and chairs as lily pads; or (iv) write a computer program.

Now let's analyse this problem. The red frog can jump to the middle. As a result, the blue frog can jump over the red frog to the end lily pad. Then the red frog can finish the swap by jumping to the empty right lily pad. That's **three** moves altogether.

Note. Is this the only way to swap the frogs? Is it possible to complete the swap in two jumps? Can the swap be done in four jumps?

- b. Suppose there are now five lily pads, two red frogs and two blue frogs. Their positions are shown in Diagram 2. The moves are the same. What is the **smallest** number of jumps needed to swap the red frogs to where the blue frogs started, and the blue frogs end up where the red frogs started?

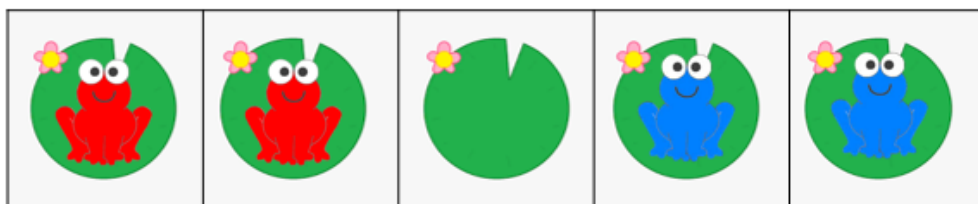


Diagram 2. Four frogs and five lily pads.

What we'll do now is to write down possible moves and see what happens. To be able to keep track of the individual frogs, we introduce some new notation (symbols). Let's say the initial position is

$$R_1 R_2 - B_1 B_2$$

Where R represents a red frog, B a blue frog and – an empty lily pad. With any luck we have a symmetrical situation and whatever happens after moving the right red frog, R_2 , will give the same number of moves as if we moved the leftmost blue frog B_1 first. Oh, of course. We forgot about the leftmost red frog R_1 . It might have the first jump by going over R_2 . (And so on for B_2 .) We have two cases, so we'll break the working up into two cases, carefully called Case 1 and case 2!

Case 1: Move the R_2 first.

Move 1: $R_1 - R_2 B_1 B_2$.

From here, we have possible 2nd moves, move R_1 or B_1 . We show R_1 moving first.

Possible Move 2: $- R_1 R_2 B_1 B_2$.

This is a bit of a problem. Only the red frogs can move, and they have to move backwards! If we do this a red frog is going to have to move back and that's a waste of two moves.

So it looks as if we should move the leftmost B. And here it is.

Better Move 2: $R_1 B_1 R_2 - B_2$.

From here there are again two possible jumps, R_2 to move right or B_2 to move left.

Possible Move 3: $R_1 B_1 - R_2 B_2$.

Can you see a difficulty with this? If we move R_1 it will give us an $R_1 R_2 (- B_1 R_1 R_2 B_2)$ and although one B_1 can move, B_2 is stuck. A similar problem happens if we move B_2 to the left.

It turns out that every time we move two frogs of the same colour together something gets stuck. This means that we only want frogs of the same colour together when they are in their final positions.

Moving R_2 right didn't work, so let's move B_2 left.

Better Move 3: $R_1 B_1 R_2 B_2 -$.

Now we can jump the red frogs over.

Move 4: $R_1 B_1 - B_2 R_2$ and Move 5: $- B_1 R_1 B_2 R_2$.

We're almost there now...

Move 6: $B_1 - R_1 B_2 R_2$. Move 7: $B_1 B_2 R_1 - R_2$ and Move 8: $B_1 B_2 - R_1 R_2$

Case 2: Move the leftmost Red frog.

This gives us

$- R_2 R_1 B_1 B_2$

Didn't that cause us enough problems in Case 1. It causes us to waste jumps. So let's forget it.

Great, so we can move the frogs in **eight** moves.

Notes.

- (i) What is the fewest number of moves that you need here? Can you **prove** it to anyone else?
- (ii) What situations are problematic and hold up your rush of frogs? Can you always avoid these positions?
- (iii) Can you always make sure that a frog never goes backwards?
- (iv) Why is part b. an extension of part a.?

(v) If you were teaching a friend how to do this, can you think of three or four hints that you might need to give them?

- c. Suppose there are now seven lily pads, three red frogs and three blue frogs. Their positions are shown in Diagram 3.

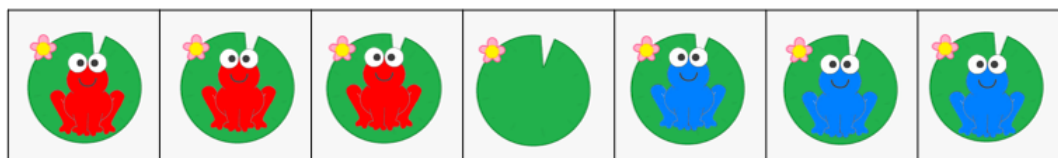


Diagram 3. Six frogs and seven lily pads.

Before you start this situation, try to guess how many moves the frogs might need. Record your guess or guesses. We think you should manage the jumping interchange in **15** moves. Do your jumps add up to that? Did you avoid lots of problems?

Here's one way to get 15 moves:

$R_1 R_2 R_3 - B_1 B_2 B_3;$

$R_1 R_2 - R_3 B_1 B_2 B_3; \quad R_1 R_2 B_1 R_3 - B_2 B_3; \quad R_1 R_2 B_1 R_3 B_2 - B_3; \quad R_1 R_2 B_1 - B_2 R_3 B_3;$

$R_1 - B_1 R_2 B_2 R_3 B_3; \quad - R_1 B_1 R_2 B_2 R_3 B_3; \quad B_1 R_1 - R_2 B_2 R_3 B_3; \quad B_1 R_1 B_2 R_2 - R_3 B_3;$

$B_1 R_1 B_2 R_2 B_3 R_3 -; \quad B_1 R_1 B_2 R_2 B_3 - R_3; \quad B_1 R_1 B_2 - B_3 R_2 R_3; \quad B_1 - B_2 R_1 B_3 R_2 R_3;$

$B_1 B_2 - R_1 B_3 R_2 R_3; \quad B_1 B_2 B_3 R_1 - R_2 R_3; \quad B_1 B_2 B_3 - R_1 R_2 R_3.$

Notes.

(i) Did you guess 15. Well done, but how did you manage to get the right answer?

(ii) Did you avoid the tricky situation?

(iii) Do you see the process of setting up alternate Red, or Blue, frogs so that as soon as there is one spare space, your Reds will move along the chain one after the other.

(iv) Can you prove that 15 is the right answer?

(v) Can you see a pattern to the number of moves as you use more and more frogs? Let's put them in a table to help you see...

Frogs on each side	1	2	3
Minimum number of moves	3	8	15

(vi) What hints did you think of to help a friend do this? We suggest: keep the blue frogs moving to the left and the red frogs moving to the right; avoid having two reds or two blues together unless it is at the very end; and keep a record of your moves (some students see how to do it efficiently using manipulatives, but by then they have forgotten how they did it).

- d. Suppose there are now nine lily pads, four red frogs and four blue frogs.

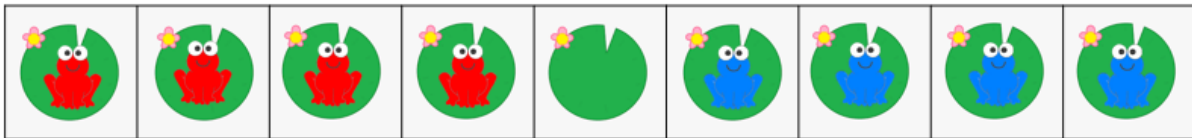


Diagram 4. Eight frogs and nine lily pads.

Looking at the table below, you may notice that the number of jumps increases by 5 and then increases by 7. You may then wonder, if this answer will be 9 bigger than the previous. This gets us to 24 jumps for four frogs on each side. Alternatively, you may have noticed that the number of jumps is always a multiple of the number of frogs. We show this in the table below:

Frogs on each side	1	2	3
Minimum number of moves	$1 \times 3 = 3$	$2 \times 4 = 8$	$3 \times 5 = 15$

With the green numbers increasing by 1 each time, this suggests that when we have 4 frogs we might need to calculate 4×6 , again this gives 24 and now 24 is looking most promising!

Working it through using the rules reds move right and blues move left and never having two of the same colour together you should get **24** (it may take a few goes to get all movements and counting correct, don't lose hope!).

- e. Predict and then check the smallest number of jumps for 5 frogs. Prove that your answer is correct.

Following the patterns found above, we believe there will be $5 \times 7 = 35$ moves for this problem. Again, we can work it through using our rules and see that this is correct, five frogs on each side can change places in **35** moves.

Note. While we have seen some lovely patterns in the numbers so far, we have relied entirely on "acting it out" to check if each value is indeed the minimum number of moves. With five cases that work, do we have enough information to say that this pattern will hold forever? Unfortunately, the answer is 'absolutely not!' [Consider for example the polynomial $n^2 - n + 41$. For $n = 1, 2, 3$ and 4 and LOTS more values too, a prime number is produced, though eventually you'll find that there is a value of n that doesn't give a prime number. Can you find it?]

Lucky we are about to be challenged to consider the general case. If we can verify our formula we can then use it to find the minimum number of moves it will take to move any number of frogs – cool!!

Further explorations: What if there were n frogs of each colour? What is the smallest number of jumps that the frogs would need to take. What if there were n frogs of one colour and m frogs of another??

These are **generalisations** of the original problem. We are trying to find a formula that can tell us the minimum number of moves for any number of frogs.

Let's first consider the case where we have n frogs of each colour.

Based on the patterns we've seen so far, we make a prediction in the last column of our table.

Frogs on each side	1	2	3	4	5	n
Minimum number of moves	$1 \times 3 = 3$	$2 \times 4 = 8$	$3 \times 5 = 15$	$4 \times 6 = 24$	$5 \times 7 = 35$	$?? \ n \times (n+2) \ ??$

Okay, let's think more about what happened in the case where we had three frogs on each side. Below shows the frogs initial position and a possible final position. Other final positions are possible, though we are going to consider this one in detail first.

Initial Position	R ₁	R ₂	R ₃		B ₁	B ₂	B ₃
Possible Final Position	B ₁	B ₂	B ₃		R ₁	R ₂	R ₃

To get from their initial to final positions, we see that each red frog has overall moved 4 places right and each blue frog has moved 4 places left. This gives us a total of $3 \times 4 \times 2 = 24$ moves.

In the general case we have:

Initial Position	R ₁	R ₂	...	R _n		B _n	...	B ₂	B ₃
Possible Final Position	B ₁	B ₂	...	B _n		R _n	...	R ₂	R ₃

Here we have n red frogs that have to move $n+1$ places each and n blue frogs that have to move $n+1$ places each. This gives us a total of $n \times (n+1) \times 2$ places.

Every move a frog makes is either a *slide* across or a *jump* over. Let's represent the number of slides by the letter s and the number of jumps by the letter j . A slide moves one place and a jump moves 2 so we know that $s + 2j = n \times (n+1) \times 2$.

Let's verify this formula for the case where we have our three frogs on each side. Below we highlight a frog that has completed a slide in blue and a frog that has completed a jump in red.

R₁ R₂ R₃ – B₁ B₂ B₃;

R₁ R₂ – **R₃** B₁ B₂ B₃; R₁ R₂ **B₁** R₃ – B₂ B₃; R₁ R₂ B₁ R₃ **B₂** – B₃; R₁ R₂ B₁ – B₂ **R₃** B₃;

R₁ – B₁ **R₂** B₂ R₃ B₃; – **R₁** B₁ R₂ B₂ R₃ B₃; **B₁** R₁ – R₂ B₂ R₃ B₃; B₁ R₁ **B₂** R₂ – R₃ B₃;

B₁ R₁ B₂ R₂ **B₃** R₃ –; B₁ R₁ B₂ R₂ B₃ – **R₃**; B₁ R₁ B₂ – B₃ **R₂** R₃; B₁ – B₂ **R₁** B₃ R₂ R₃;

B₁ **B₂** – R₁ B₃ R₂ R₃; B₁ B₂ **B₃** R₁ – R₂ R₃; B₁ B₂ B₃ – **R₁** R₂ R₃.

Here we have $s = 6$ and $j = 9$, and indeed $s + 2j = 6 + 2 \times 9 = (3 + 1) \times 3 \times 2$.

Now for our frogs to swap sides, each red frog needs to cross over each blue frog and these cross overs can only happen with a jump. Can you see how many jumps you will need to occur in general with n frogs? There are n frogs that need to cross over n frogs so that requires $n \times n = n^2$ jumps.

We can now re-write our formula

$$s + 2j = n \times (n + 1) \times 2$$

as

$$s + 2n^2 = n \times (n + 1) \times 2$$

$$s + 2n^2 = (n^2 + n) \times 2$$

$$s + 2n^2 = 2n^2 + 2n$$

$$s = 2n$$

Now we have a rule for the number of jumps and the number of slides. The total number of moves is $s + j = 2n + n^2 = n \times (n+2)$. Woooo hoooo!!! We did it!! Now, there is actually another piece of the puzzle to consider – do we actually have a process for swapping the frogs over in this number of moves? To prove this, we need to use a technique called mathematical induction. This is quite an advanced technique that will be introduced a little later on. For those interested, a full proof for this problem can be found in Derek's book "[Problem Solving – The Creative Side of Mathematics](#)"

What if there were m frogs on one side and n on the other? What is the smallest number of jumps needed here?

We can approach this in the same way as above.

- The red frogs will each need to move $n+1$ places, and the blue frogs will need to move $m+1$ places. This gives a total of $m(n+1)+n(m+1)$ places.
- For the red and blue frogs to swap places mn crossings will need to occur, so $j = mn$
- Now we have $s+2j=m(n+1)+n(m+1)$ which can be re-written as $s+2mn=m(n+1)+n(m+1)$. Similar algebraic steps show that $s=m+n$.
- $s+j=m+n+mn$

It is recommended that you check this out by testing out a few cases, such as $m=1$, $n=2$ and $m=4$ and $n=2$.

Note. Something else you might like to consider is starting with the empty lily pad on the left. How does this change the minimum number of jumps?

Problem 4

- a. Consider the 3x3 grid shown below.
- How many 1x1 squares are on this grid?
 - How many 2x2 squares are on this grid?
 - How many 3x3 squares are on this grid?

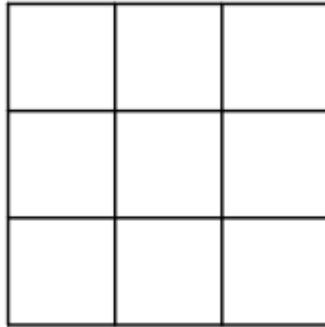
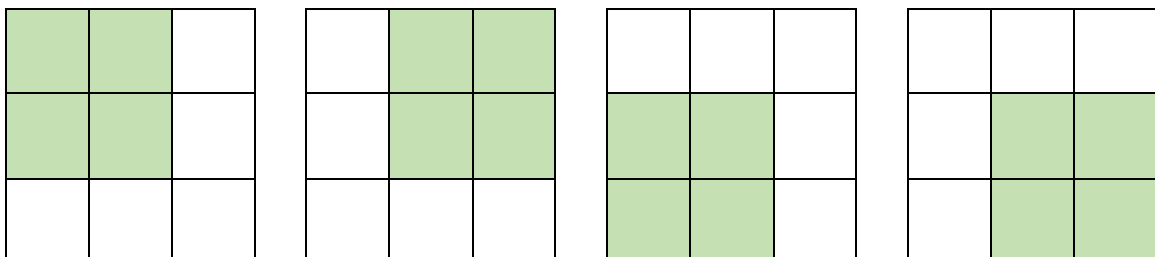
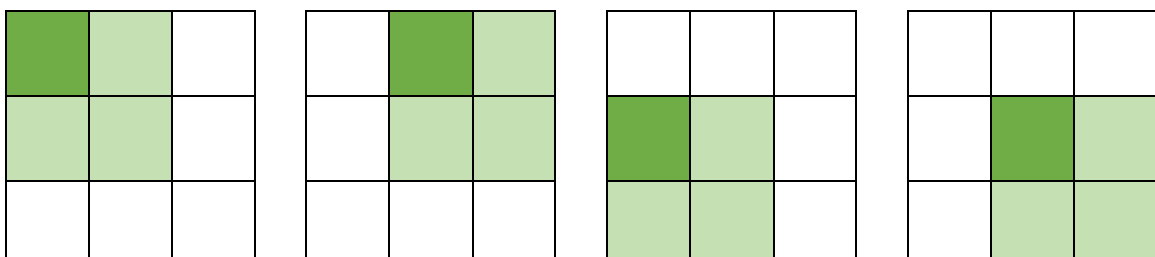


Diagram 1. A 3 x 3 grid.

- All of the 1 x 1 squares are clearly shown so all you have to do is count them. There are nine.
- This is slightly more difficult. The diagrams below show all options, and we see that there are four. Though how can we work systematically to make sure we don't miss any and also make them easier to count?

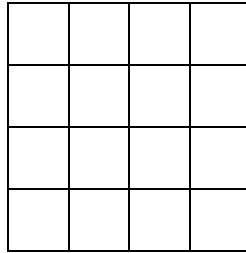


Well, we can highlight the top left square of each and consider each square of the grid starting at the top left and working our way across as far as we can and then work through each row below in the same way.

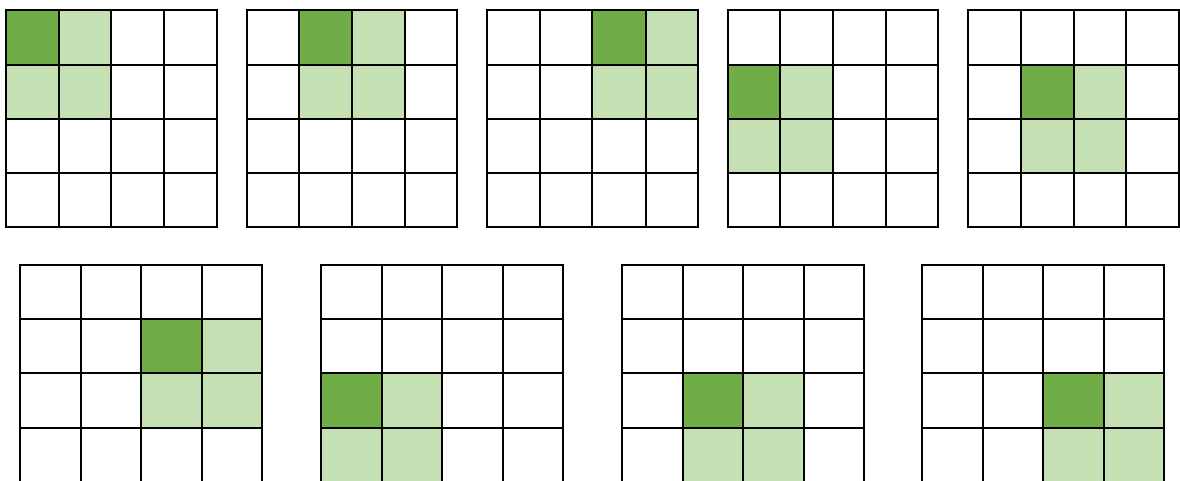


- Things get easier again here, the 3 x 3 grid is also a 3 x 3 square. As a result, there is only one.
- b. Consider a 4x4 grid.
- How many 1x1 squares are on this grid?
 - How many 2x2 squares are on this grid?
 - How many 3x3 squares are on this grid?
 - How many 4x4 squares are on this grid?

Let's again draw out a diagram so we've got something to work with.

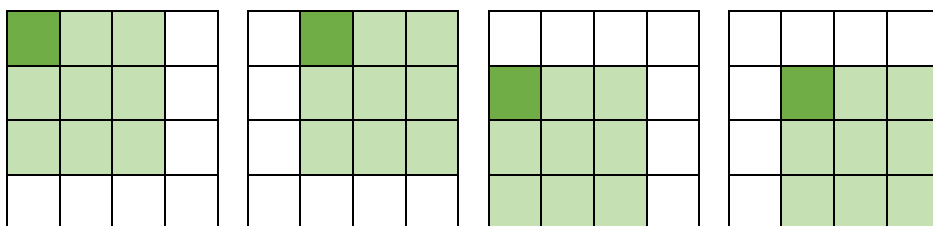


- i. Again, all of the 1×1 squares are clearly shown we count them and get 16.
- ii. Time for our diagrams, though this time let's go straight into highlighting the top left square and working in our systematic way.



We see we have nine 2×2 squares in our 4×4 grid.

- iii. We follow a similar approach for our 3×3 squares.



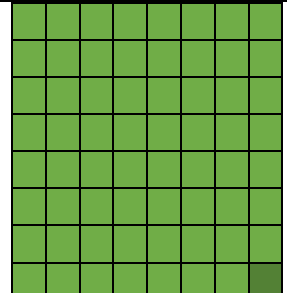
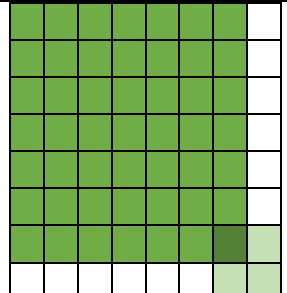
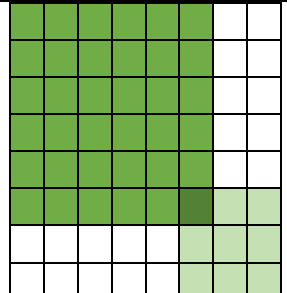
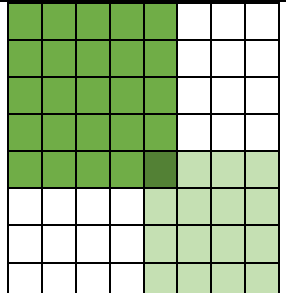
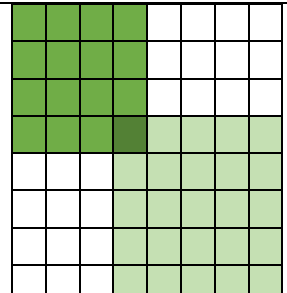
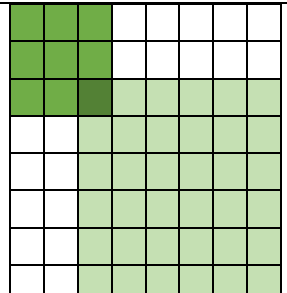
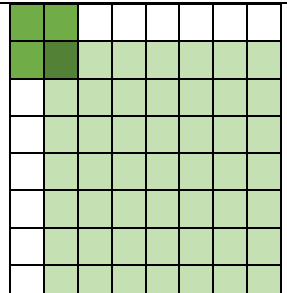
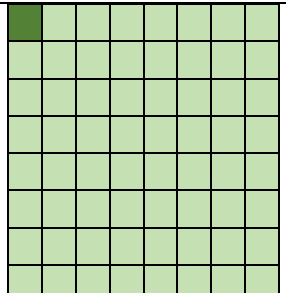
The answer is four.

- iv. This answer is the easiest of the four parts, so we won't embarrass you by saying how we got one.

Note. There is something special about the answers we have found in all of the parts of both a and b. What is it and why is it what it is?

- c. It was once claimed that there are 204 squares on an ordinary chessboard. Can you justify this claim?

Okay, so drawing out all the diagrams for all the possible squares on an 8x8 grid would be incredibly time consuming. Fortunately, we can simplify our diagrams by just showing the top left square of each smaller square. We find that it also helps if we shade in the bottom right square in its entirety to remind us of the size and make sure it fits. Let's go!

			
8 x 8 = 64 1x1 squares	7 x 7 = 49 2x2 squares	6 x 6 = 36 3x3 squares	5 x 5 = 25 4x4 squares
			
4 x 4 = 16 5x5 squares	3 x 3 = 9 6x6 squares	2 x 2 = 4 7x7 squares	1 x 1 = 1 8x8 square

And so we have $8^2 + 7^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2 = 204$ squares. Fabulous!!










Note. Do you know how to add up the squares of the first n numbers without calculating each individual square? We're going to delay things until we've had a close look at Problem? 5. But we challenge you to try for yourself. You might want to know that $(n + 1)^3 - n^3 = 3n^2 + 3n + 1$.

Further exploration: How many rectangles are on a chessboard?

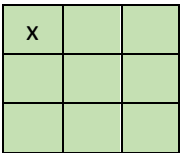
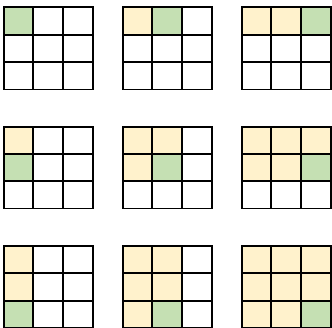
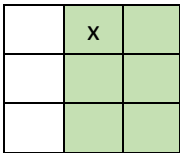
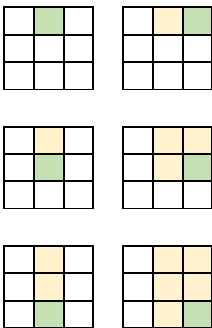
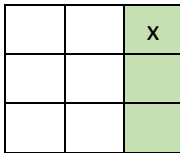
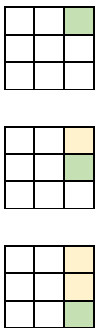
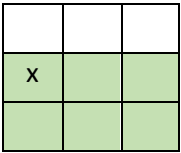
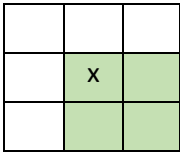
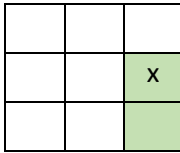
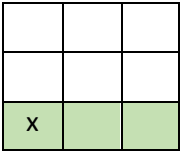
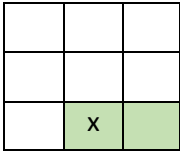
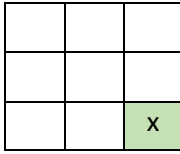
First of all, we should note that a square is a rectangle. Now, we will need to work systematically as there are A LOT of rectangles. One approach is to again consider what happens if any given singular square is the top left square of a rectangle.

Before we move onto the 8x8 case, let's examine a smaller case so we can include all the details.

To be able to talk about singular squares, let's introduce a coordinate system.

	A	B	C
1			
2			
3			

We will use an x to mark the top left square of a rectangle and highlight in green all possible squares that could be the bottom right corner of the rectangle. To give this further clarity, for the first row we show all rectangles. Note that we keep the other squares within the rectangle yellow to show how it is the bottom right square that continues to change location.

<div> <div> <div>A</div> <div>B</div> <div>C</div> </div> <div> <div>1</div> <div>2</div> <div>3</div> </div> </div>  <p>3 x 3 = 9 rectangles starting at A1</p> 	<div> <div> <div>A</div> <div>B</div> <div>C</div> </div> <div> <div>1</div> <div>2</div> <div>3</div> </div> </div>  <p>3 x 2 = 6 rectangles starting at B1</p> 	<div> <div> <div>A</div> <div>B</div> <div>C</div> </div> <div> <div>1</div> <div>2</div> <div>3</div> </div> </div>  <p>3 x 1 = 3 rectangles starting at C1</p> 
<div> <div> <div>A</div> <div>B</div> <div>C</div> </div> <div> <div>1</div> <div>2</div> <div>3</div> </div> </div>  <p>2 x 3 = 6 rectangles starting at A2</p>	<div> <div> <div>A</div> <div>B</div> <div>C</div> </div> <div> <div>1</div> <div>2</div> <div>3</div> </div> </div>  <p>2 x 2 = 4 rectangles starting at B2</p>	<div> <div> <div>A</div> <div>B</div> <div>C</div> </div> <div> <div>1</div> <div>2</div> <div>3</div> </div> </div>  <p>2 x 1 = 2 rectangles starting at C2</p>
<div> <div> <div>A</div> <div>B</div> <div>C</div> </div> <div> <div>1</div> <div>2</div> <div>3</div> </div> </div>  <p>1 x 3 = 3 rectangles starting at A3</p>	<div> <div> <div>A</div> <div>B</div> <div>C</div> </div> <div> <div>1</div> <div>2</div> <div>3</div> </div> </div>  <p>1 x 2 = 2 rectangles starting at B3</p>	<div> <div> <div>A</div> <div>B</div> <div>C</div> </div> <div> <div>1</div> <div>2</div> <div>3</div> </div> </div>  <p>1 x 1 = 1 rectangle starting at C3</p>

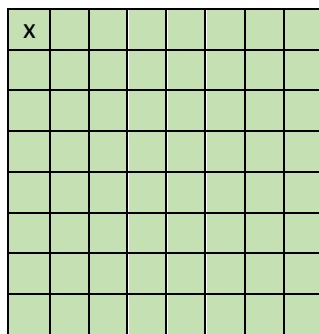
We can add these up to see that there are 36 different rectangles in the 3x3 grid.

Let's now return to our 8x8 chessboard.

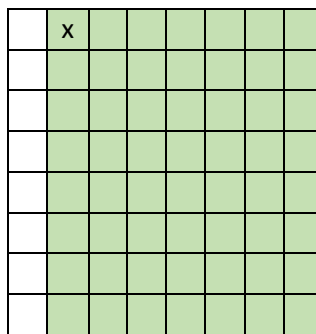
	A	B	C	D	E	F	G	H
1								
2								
3								
4								
5								
6								
7								
8								

We will again use an x to mark the top left square of a rectangle and highlight in green all possible squares that could be the bottom right corner of the rectangle.

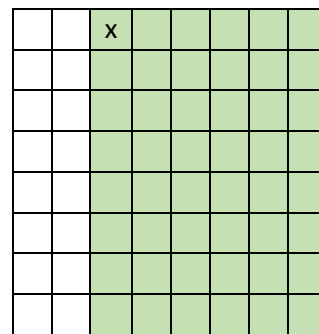
Let's start by considering the first four marked squares.



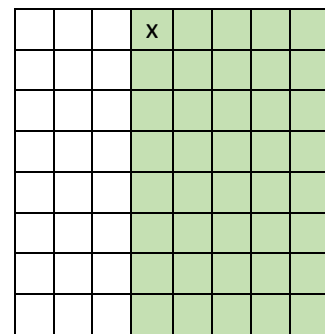
$8 \times 8 = 64$ rectangles
starting at A1



$8 \times 7 = 56$ rectangles
starting at B1



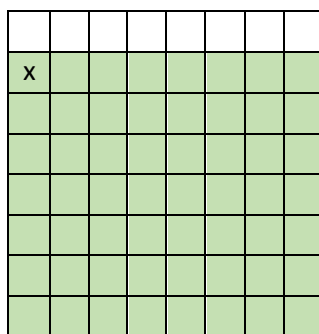
$8 \times 6 = 48$ rectangles
starting at C1



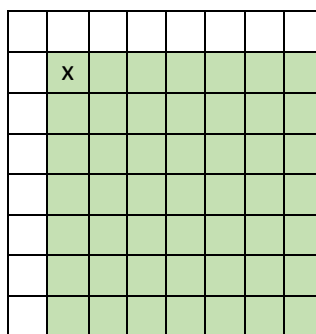
$8 \times 5 = 40$ rectangles
starting at D1

Working along the whole top row we have $8 \times 8 + 8 \times 7 + 8 \times 6 + \dots + 8 \times 1 = 8 \times (8 + 7 + 6 + 5 + 4 + 3 + 2 + 1) = 8 \times 36 = 288$

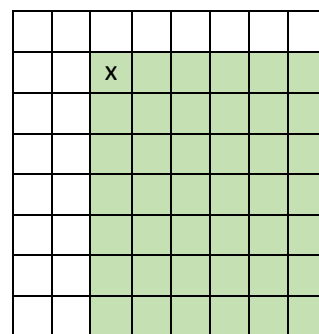
What about the second row? Well, a similar thing will happen, except now we'll have only 7 rows.



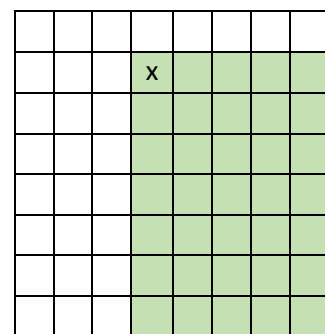
$7 \times 8 = 56$ rectangles
starting at A2



$7 \times 7 = 49$ rectangles
starting at B2



$7 \times 6 = 42$ rectangles
starting at C2



$7 \times 5 = 35$ rectangles
starting at D2

So the second row has $7 \times 8 + 7 \times 7 + 7 \times 6 + \dots + 7 \times 1 = 7 \times (8 + 7 + 6 + 5 + 4 + 3 + 2 + 1) = 7 \times 36 = 252$.

Now this pattern will continue for every row so we will have 6×36 for the third row down to 1×36 for the eight.

This gives a total of $8 \times 36 + 7 \times 36 + 6 \times 36 + \dots + 1 \times 36 = 36 \times (8 + 7 + 6 + 5 + 4 + 3 + 2 + 1) = 36 \times 36 = 1296$ rectangles on a chess board.

Note. The following problem will empower you to find a general formula for the number of rectangles on any size square grid, and with a bit of extra work you'll be able to generate a formula for the number of the squares on any sized grid too!

Problem 5

- a. There are four discs on the bottom of this triangle. How many discs are there altogether?



Diagram 1. How many discs?

Hopefully it isn't too hard to count the discs. We got 10.

- b. How many discs would there be altogether if there were 10 discs on the bottom?

Incidentally, this problem is an **extension** of the problem in a. Extensions are problems that look like another one, but there is something small changed. In this case the 4 is changed to a 10, but the overall ideas are the same.

The simplest way to do this is to draw out the disc picture where there are 10 discs on the bottom, then count. It looks as if the answer is 55.

But if adding up that many numbers isn't all that appealing, there are other, more efficient ways to look at things. Let's look back on Diagram 1 above. Counting the discs here can be done by noticing that the top row of discs added to the bottom row of discs gives 5 discs. And the second top row plus the second bottom row gives 5 as well. (Hmm, that's interesting!) So, we can get 10 discs quickly by just adding the 5 and the 5.

Looking at the triangle with 10 discs on the bottom row, we get 11 (from the top plus the bottom); another 11 (from the second top row added to the second bottom row); a further 11 (from the third top row added to the third bottom row); and 11 more (from the fourth top row added to the fourth bottom row); and finally, 11 extras (from the fifth top row added to the fifth bottom row). Since there are five lots of 11 there must be $5 \times 11 = 55$ discs. That was quicker, and I'm sure much more satisfying too!

- c. How many discs would there be altogether if there were 100 discs on the bottom?

This problem is an extension of the one in a. It's also an extension of the problem in b.

We suggest that you **don't** draw a triangle of discs with 100 on the bottom row. The little idea we used in b. could work for you though. So top and bottom row discs add together to give 101. The question now is how many pairs of rows add to 101? Row 1 and 100 go together; row 2 and row 99 go together; row 3 and 98 go together; row 4 and row 97 go together; ... But where does it all end? Surely when you get to the middle. What are the numbers of the rows here? How do 50 and 51 sound.

If that is true, the answer should be $50 \times 101 = 5050$. Did you get 5050? Did you use a different way to ours?

Note. We are a little worried at this point. Luckily for us 4 and 10 are even, so we can pair up rows as we have done in a. and b. But what if we wanted to know the number of discs in a triangle with 5 rows and there are 5 discs on the bottom row? How would we deal with this? Try the 5 row problem and see what you get.

- d. How many discs would there be altogether if there were n discs on the bottom? This is called a **generalisation** of the problem with 4, 10 and 100 rows. It is a generalisation because it looks at every possible case built on the triangles of parts a., b. and c.

There are two things we want to do here. First, use the ideas we have been employing above to guess the answer. (Actually, mathematicians are too sophisticated to guess. Instead, they **conjecture**. Which is a guess. Oh well.)

Now if $n = 4$, we got $5 \times (4 \div 2)$, 5 for the addition of two symmetric rows and half of the number of rows for the number of the 5 that you get.

And if $n = 10$, we got $11 \times (10 \div 2)$, 11 for the addition of two symmetric rows and half of the number of rows for the number of the 5 that you get.

If we are lucky, for n rows, we might get $(n+1) \times (n \div 2) = (n+1)n/2$. This is our conjecture.

Can we prove that $(n+1)n/2$? It might just be worth checking that $n = 5$ satisfies our conjecture (remember, we were a little worried about cases where the number of rows were odd). Does it?

Whether or not you got the case for $n = 5$ working, let's just plough on. Now we know three ways to make progress on this. So here we go.

Proof 1: Let

$$S = 1 + 2 + 3 + \dots + (n - 2) + (n - 1) + n.$$

This is the sum we want. If that worries you then let $n = 4$ or 10 as checks. (In fact for anything we say in this proof, if you are not sure, substitute 4 or 10 or both for n and see how it all turns out.)

Now let's turn things around. It should be clear that

$$S = n + (n - 1) + (n - 2) + \dots + 3 + 2 + 1.$$

Having got this far add the numbers that lie one above the other.

So $2S = [1 + n] + [2 + (n - 1)] + [3 + (n - 2)] + \dots + [(n - 2) + 3] + [(n - 1) + 2] + [n + 1].$

Now all the square brackets add to $n + 1$. (Does that ring any bells from $n = 4$ and 10?) And the number of square brackets is n . That gives us

$$2S = (n + 1)n. \quad \#$$

So S really is $(n+1)n/2$. Bingo!

No doubt you've already worked through it, though let's make sure this works for $n = 4, 5$ and 10.

n	discs	formula
4	10	$\frac{(4 + 1)4}{2} = \frac{5 \times 4}{2} = 10$

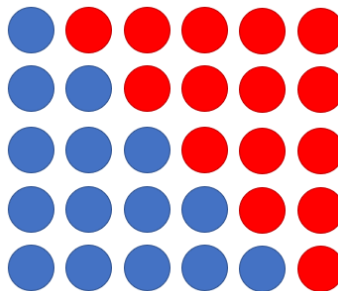
5	15	$\frac{(5+1)5}{2} = \frac{6 \times 5}{2} = 15$
10	55	$\frac{(10+1)10}{2} = \frac{11 \times 10}{2} = 55$

Proof 2. The second proof is very similar to the first, though we approach it visually rather than algebraically.

Let's start considering the case where $n = 5$. Here we are required to count the number of discs in the following diagram. Again, we will use the letter S to represent the total number of discs.



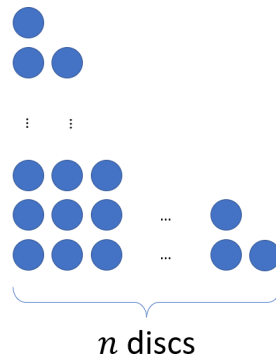
Instead of just counting these discs, let's first create a rectangle by building an identical stack, however made with red discs so it is easily distinguished from the blue, and putting it next to the blue stack.



The total number of discs in this diagram is $2S$. The total number of discs is also 5×6 since there are 5 rows of 6 discs. Since $2S = 30$, $S = 15$. Beautiful!

Now how does this work in general??

In general, we start off with a triangle that has one disc at the top and n discs at the bottom.



Now look at Diagram 3, where again we've placed a second stack (identical in number, just coloured red and rotated) next to the blue stack.

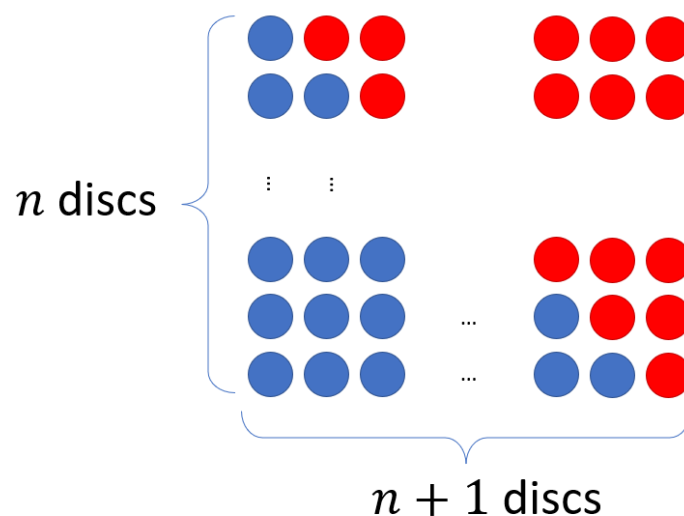


Diagram 3. The situation with two triangles, but with one triangle rotated.

Counting the discs in the $(n + 1) \times n$ rectangle gives $(n + 1)n$ discs. Again the original triangle contains half of the discs as the rectangle does. So the number of discs in the triangle is $(n+1)n/2$. #

Introduction to Proof 3.

As mentioned earlier, Proofs 1 and 2 are really the same way of getting the answer. The next proof is quite different, however. It uses a technique called Mathematical Induction. It works this way:

Step 1: Make sure that an early case checks with the formula, say $n = 1$.

Step 2: Assume, that the formula is true for some bigger number, $n = k$, say.

Step 3: Prove that the formula is true for $n = k + 1$.

Then this is how the proof works. If $n = k$ is true, then $n = k + 1$ is true if we can do Step 3. But then if $k = 1$, the formula is true because that is where we started. Then put $k = 1$ in Step 2. From Step 3 we know that the formula is true for $k + 1 = 1 + 1 = 2$.

But then if $k = 2$ is then put $k = 2$ in Step 2. From Step 3 we know that the formula is now true for $k + 1 = 2 + 1 = 3$.

But then if $k = 3$ is true then put $k = 3$ in Step 2. From Step 3 we know that the formula is true for $k + 1 = 3 + 1 = 4$.

Do this again and again until you have got to the number you want, $n = 100$, say. But this method will show that you can eventually get to any number n and the formula is proved for any n .

Now let's see this in action.

Proof 3. Step 1. Is the formula true for $n = 1$? Well clearly if there is only one row to the triangle there is only one disc. If we put $n = 1$ into the formula

$$S = (n + 1)n/2$$

$$\text{Then } S = (1 + 1) \times 1/2 = 1.$$

The formula is correct for $n = 1$ at least.

Step 2. Assume that for a triangle with k rows there are $(k + 1)k/2$ discs.

Step 3. Suppose that we have $k + 1$ rows. Then the triangle with $k + 1$ rows will have as many as there are in a triangle with k rows, plus the last row with $k + 1$ discs in.

So the sum for the triangle with $k + 1$ rows is

$$S = (k + 1)k/2 + (k + 1) = (k + 1)k/2 + 2(k + 1)/2 = (k^2 + k)/2 + (2k + 2)/2 = (k^2 + 3k + 2)/2 = (k + 2)(k + 1)/2 = (k + 1 + 1)(k + 1)/2.$$

And this is the formula when $n = k + 1$.

Hence by the method of Mathematical Induction, the proof is complete. #

Proof 4. Yes we did say we had three proofs, but we've just thought of another one!

It turns out that $(x + 1)^2 = x^2 + 2x + 1$. So $(x + 1)^2 - x^2 = 2x + 1$. Now watch this.

$$(n + 1)^2 - n^2 = 2n + 1 \text{ (letting } x = n)$$

$$n^2 - (n - 1)^2 = 2(n - 1) + 1 \text{ (letting } x = n - 1)$$

$$(n - 1)^2 - (n - 2)^2 = 2(n - 2) + 1$$

...

$$3^2 - 2^2 = 2(2) + 1$$

$$2^2 - 1^2 = 2(1) + 1$$

When we add everything on the left of the equal signs a lot of the terms disappear because they are on two levels, first as positives and then as negatives. In the end we only get $(n + 1)^2 - 1^2$ on the left.

On the right though we first get $2[n + (n - 1) + (n - 2) + \dots + 2 + 1]$ which is twice the sum of the first n numbers – twice the sum that we're trying to find – along with n ones.

Combining both the left and the right additions we find that

$$(n + 1)^2 - 1^2 = 2[n + (n - 1) + (n - 2) + \dots + 2 + 1] + n.$$

Tidying all of this up with a little algebra we get ... Of course! Wow!

#

Further exploration: Consider triangles with rows containing only even numbers of discs what happens now? What if instead only odd numbers of discs were used? What happens if you extend the idea to trapezium. How many discs would there be in a trapezium with 5 rows at the top and 10 at the bottom? What other patterns could you explore?

Now you may come to an immediate guess or you may have no clues about any of these. No matter. In both cases it's good to try a few small examples as we did with 4 and 10 in parts a. and b. If you have a guess then you might find that the examples substantiate your conjecture. If you have no clue then there's a chance that the examples will give you some inspiration.

Another technique is to change the problem into something that you already know the value of. From there, maybe you can count the addition or subtraction of the pieces that change the new problem back to one you know how to solve.

Anyway, there are a few ways to answer the question about even numbers. Perhaps the easiest way is to divide the number of discs in every row by 2. Find the sum for the number discs for the original problem and then multiply by 2 to give the final answer. And letting $n = 2m$ in the formula might be useful to. Or ...

In the odd problem you are looking at triangles that start from the top with rows of 1 disc, 3 discs, 5 discs and so on. Again there are a variety of ways of tackling this problem. How about add one disc to each row? We have now turned the problem into a problem we know how to solve. All we have to do next is to take off all the added 'one discs'. So we have a formula we know minus n discs. That might factor nicely into something quite simple.

What can you add to the trapezium to get back to the original triangles? Again, you might head in the way of the problem we proved in part d.

There are a large number of possibilities for you to explore in the last question above. You could extend the problem by changing the numbers for example. We've asked you about odd and even numbers of discs, what about any multiple, or sequences like 3, 7, 11, 15, ... You can also extend the original problem by changing the triangle to any shape you like.