

## Problem 6

a. Using only 5¢ and 10¢ coins, in how many ways can you make 20¢?

You can attempt to solve this problem just writing down options as they come to you, two lots of 10c, 4 lots of 5c etc. However, how will you ever be sure you have them all?

So, just as we did with Leo in Problem 1, being systematic (working in some order) is a great idea. The two obvious ways to be systematic here are to (i) first start with two of the 10¢ coins, then just one 10c coin, and then no 10¢ coins; or (ii) to start with the 5¢ coins in a similar way. Of course, a good way to check your answer is to do the question both ways and make sure that you have the same number and arrangements each time.

We'll use method (i) because the 10¢ coins are the biggest part of the 20¢ and so will use up most of the 20¢. That will leave less choice for the 5¢.

- Two 10¢ coins:  $2 \times 10¢ = 20¢$ . There is clearly only one way to use two 10¢ coins.
- One 10¢ coins:  $1 \times 10¢ = 10¢$ . This leaves 10¢ to make up. With 5¢ coins there is only one way to do this ( $2 \times 5¢$ ).
- No 10¢ coins: Now we have to make up 20¢ with 5¢ coins. This can only be done in one way.

We see that there are **three** ways to make 20¢ using 5¢ and 10¢ coins.

b. Using only 5¢ and 10¢ coins, in how many ways can you make 50¢?

Being systematic is the way to go again.

- Five 10¢ coins:  $5 \times 10¢ = 50¢$ .
- Four 10¢ coins:  $4 \times 10¢ = 40¢$  and  $2 \times 5¢ = 10¢$ .
- Three 10¢ coins:  $3 \times 10¢ = 30¢$  and  $4 \times 5¢ = 20¢$
- Two 10¢ coins:  $2 \times 10¢ = 20¢$  and  $6 \times 5¢ = 30¢$ .
- One 10¢ coin:  $1 \times 10¢ = 10¢$  and  $8 \times 5¢ = 40¢$ .
- No 10¢ coins:  $10 \times 5¢ = 50¢$ .

We see that there are **six** ways to make 50¢ using 5¢ and 10¢ coins.

It's worth just looking at Leo's problem and the 5¢ and 10¢ problem. Why don't they both give the same numerical answers? What is different about the two problems?

c. Using only 5¢ and 10¢ coins, in how many ways can you make 100¢?

If we can see the pattern of ways, then we might be able to guess, sorry conjecture, the number of ways here:

- for 20¢ we got 3 ways, and
- for 50¢ we get 6 ways.

Does it sound likely that the answer is always one more than the number of tens of 10¢'s? Does this answer make logical sense? Well, yes, it does. In each case we can have anywhere from 0 up to the number of 10¢ that fit in. So here we have  $10+1=11$ . The answer is **11**.

d. Using only 5¢ and 10¢ coins, in how many ways can you make  $100n$  ¢?

Take any number of 10¢ pieces you like. What's left is going to be  $100n \text{ ¢} - 10\text{¢}$ . Since 5¢ goes into that  $100n \text{ ¢} - 10\text{¢}$  then there is one way for each number of 10¢ coins you choose. Now you can choose  $10n$  10¢ coins,  $10n - 1$  10¢ coins,  $10n - 2$  10¢ coins, on down to no 10¢ coins. Altogether there are  $10n + 1$  possible ways to choose a number of 10¢ coins. So the answer is  $10n + 1$ .

**Further exploration:** What happens if you have 5¢, 10¢ and 20¢ coins?

Let's consider the simplest case of creating 20¢. This solution is quite straight forward, as there is just the one additional way to create 20¢ than before (since we can create it using one lone 20¢ coin). So, we have 4 ways.

What if we tried to make 50¢? Well, we could have two, one or zero 20¢ coins for a start. For each of these solutions we can create the remaining amount using our 10¢ and 5¢ coins in many different ways.

- Two 20¢ coins = 40¢. We have 10¢ remaining. (From our previous work we know that) this can be made in two ways.
- One 20¢ coin = 20¢. We have 30¢ remaining. This can be made in four ways.
- Zero 20¢ coins = 0¢. We have 50¢ remaining. This can be made in six ways.

This gives us a total of 12 ways.

Okay, onto 100¢...

- Five 20¢ coins = 100¢. We have 0¢ remaining. This can be made in one way (by adding no other coins!).
- Four 20¢ coins = 80¢. We have 20¢ remaining. This can be made in three ways.
- Three 20¢ coins = 60¢. We have 40¢ remaining. This can be made in five ways.
- Two 20¢ coins = 40¢. We have 60¢ remaining. This can be made in seven ways.
- One 20¢ coin = 20¢. We have 80¢ remaining. This can be made in nine ways.
- Zero 20¢ coins = 0¢. We have 100¢ remaining. This can be made in eleven ways.

$1+3+5+7+9+11=36$ . There are 36 ways this can be done.

What about  $100n$  ¢? This is a great challenge!!

So first we need to work out how many 20¢ could be used.  $100n$  divided by 20 =  $5n$ . This means we can use 1, 2, 3, ...,  $5n+1$  20¢ coins (zero 20¢ coins up to  $5n$  20¢ coins). Let's reverse the order and start with the maximum number of 20¢ coins ( $5n$ ), we know that there is one way to create  $100n$  ¢ with  $5n$  20¢ pieces (as no other coins are needed). We also know that each time we take away a 20¢ coin we have an addition two ways to replace it with 10¢ coins, we either use one or two. So we start off with one way, then take off a 20¢ coin and have three ways, then take off another 20¢ coin and have five ways all the way up to the end where we have no 20¢ coins left to remove.

If we have no 20¢ coins in our collection then we have to make  $100n$  ¢ using just 10¢ and 5¢ coins. From our previous work we know that this can be done in  $10n+1$  ways.

The information collected so far is summarised in the table below.

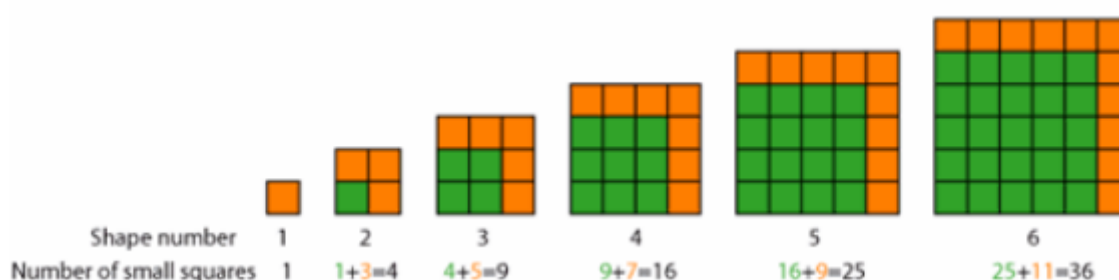
Number of 20¢ coins	$5n$	$5n-1$	$5n-2$	...	0
Number of solutions	1	3	5	...	$10n+1$

You may not be aware that the sum of odd numbers is actually a square number and this gives us a very efficient way to calculate the sum of the solutions in each case.

$2x-1$  is a formula for the odd numbers. If  $x=1$  we get 1, if  $x=2$  we get 3 etc.

To understand how the sum of odd numbers makes a square number we start with a simple case. Let's consider the first 5 odd number 1, 3, 5, 7 and 9. Here  $9=2 \times 5-1$ . The sum of these numbers is  $5^2$ .

The diagram below helps us understand why this property holds.



All that's left to do is work out which odd number  $10n+1$  is.

This requires a bit of algebra.  $10n+1=2x-1$ , so  $10n+2=2x$  and  $x=5n+1$ . And so, the sum of the solution is  $(5n+1)^2$ .

Let's check to see if this works. Well we know that if we have 100¢ we have a solution of 36. This is the  $n=1$  case, and  $(5n+1)^2=(5 \times 1+1)^2$  – awesome!!

**For fun... Here's a Further Further Exploration:** We know that there are lots of ways to create \$1 using 5¢ and 10¢ pieces. How many coins would we need to make a display showing every possible way?

So, we could use ten 10¢ coins. (That means we need to have 10 10¢ coins for a start.)

If we had nine 10¢ that way would contribute nine 10¢ coins and two 5¢ coins. Can you see a pattern?

It looks as if you will need to add from 1 to 10 inclusive to find the number of 10¢ coins that are needed. For the 5¢ coins you'll need to add all the even numbers from two to forty. Since we've given you a method before to do this kind of addition, you're pretty well finished.

There are many more ways to extend this problem too. Suppose you had twenty 5¢ coins that were all different and ten 10¢ coins that were all different, perhaps they were minted on different years. How many different ways are there to make up \$1?

Using ten 10¢ coins can only be done in one way as you just have to use all of the coins available.

Using nine 10¢ coins, you would have to leave out one 10¢ coin every time you made up the nine needed. So there are ten ways to use the 10¢ coins. Now you have to choose two 5¢ coins from the

twenty you are given. How many is that? Now multiply that number by the ten different ways to take nine 10¢ coins.

And so on and so on. Does it come to a nice answer?



## Problem 7

The pages in a book are numbered starting with page 1. One digit is used to represent the numbers from 1 to 9, whereas two digits are used to represent the numbers from 10-99 (and so on).

a. If there are 10 pages in a book. How many digits are used?

Now there are single digit numbers and 2-digit numbers on the pages of such books. The easiest way to count the digits is to go through the book and count each digit as you go. This is equivalent to noticing that the single digit numbers are 1, 2, 3, 4, 5, 6, 7, 8, 9 and the 2-digit numbers are, sorry is, 10. There are nine single digits and one 2-digit numbers, so the answer is  $9 \times 1 + 1 \times 2 = 11$ .

b. If there are 100 pages in a book. How many digits are used?

Can you see that there are nine 1-digit numbers from 1 to 9 (a total of 9 digits), ninety 2-digit numbers from 10 to 99 (nine sets of 10 groups of 2-digit numbers like 10 – 19, 20 – 29, and so on giving a total of  $9 \times 2 \times 10 = 180$  digits) and one 3-digit number, 100 (for three more digits). So altogether there are  $9 + 180 + 3 = 192$ .

c. If there are 183 pages in a book. How many digits are used?

Explain why the answer might be  $9 \times 1 + 9 \times 2 \times 10 + 84 \times 3 = 441$ .

**Further Exploration: 1.** If there are  $n$  pages in a book where  $n=1000a+100b+10c+d$ . How many digits are used?

If you are having trouble to find this, try it out with a few values of  $n$ . In other words, try a simpler case or two. Here that might be to take a specific number for  $n$ . For example, suppose  $n = 2731$ . Then you should be able to find the digits up to 999, by using the ideas of the last three questions. Now you have to count the 4-digit numbers from 1000 to 2731. Each of them has four digits and there are 2732 of them.

Note that we'll assume that  $a$  is at least 1.

The single digits here must give you 9 digits again.

The 2-digit numbers don't change, so there are 180.

For the 3-digit numbers you should notice that from 100 to 199, there are  $3 \times 100$  digits. So the hundreds numbers give  $3 \times 100 \times 9 = 2700$ .

The 4-digit numbers go from 1000 to 2731 and there are 2732 such numbers. These give  $4 \times 2732$ . The rest is easy. Just add the number of digits for the four pieces above.

In general then, you should get  $9 + 180 + 2700 + 1000a + 100b + 10c + d + 1$ , or  $2889 + 1000a + 100b + 10c + d + 1$ .

**Further Exploration: 2.** If your newspaper has 40 pages, what is the sum of the numbers on the facing middle pages? Start with a guess. Check it out. Can you prove it?

Now the sum of the numbers of the two outside pages is 41. Pull away that first double outer newspaper page from the rest of the paper. This single double page will be open at pages 2 and 39. These pages add to 41 too. Now go back to the first double page in the rest of the paper. The page numbers there are 3 and 38 to once again give 41. As you go, the small numbers get bigger one by

one and the big numbers get smaller one by one. So, on the middle page at the centre of the paper, the page numbers have to be  $20 + 21 = 41$ .

## Problem 8

Now 23 is a 2-digit number whose digits add to 5. And so are 32, and 14 and a few others.

a. How many 2-digit numbers have 6 as the sum of their digits?

First you can't use the number 6 because that has only one digit. But we could have 15, 24, 33, 42, and 51. Wait, that's not the end. We can sneak in 60. The required answer is 6.

**Additional Thought:** Hmm. Does that work for all 2-digit numbers? Is it true that there are eight 2-digit numbers whose digits add up to eight?

b. How many 3-digit numbers have 6 as the sum of their digits?

Well, the smallest number with a digit sum of 6 is 105. We can try working up "in order" from 105, like we did in part a., we get 105, 150, 203, 230, 302, 320, ... This is making me a little nervous though as I can't be sure I won't miss any. We would need to be **very** systematic to not lose a number.

Is there a better way? Well, we could first think about all the sets of 3 digits that sum to 6. Now, I first tried listing these from smallest to largest, though starting with the biggest number works better as it allows me to think of 600 and not 006. Doing this we get [6,0,0], [5,1,0], [4,2,0], [4,1,1], [3,3,0], [3,2,1] and [2,2,2]. We've been systematic here by working starting with 6, then 5, then 4 and so on. Then we would need to put the digits in different orders to get the complete list of 3-digit numbers. We show all possibilities in Table 1. Again, starting from the largest option and working our way to the smallest.

Table 1: Systematic solution to part b.

Digits	Numbers	Number of ways of arranging
600	600	1
510	510, 501, 150, 105	4
420	420, 402, 240, 204	4
411	411, 141, 114	3
330	330, 303	2
321	321, 312, 231, 213, 132, 123	6
222	222	1
TOTAL:		21

So, we see that there are 21 3-digit numbers whose digit sum is 6.

c. How many 4-digit numbers have 6 as the sum of their digits?

We can use a similar approach to systematically list the 4-digit numbers. Have a go at it yourself before looking at the table on the following page.



## 2-digit numbers

Working from smallest to largest for the first few possible sums we have: [10], [11,20], [12,21,30], [13,22,31,40]. Here, we see that we gain one more solution each time, and that the number of solutions always matches the digit sum (there are four 2-digit numbers that have a digit sum of four).

Will this pattern continue? Why/why not?

The pattern will continue because we can always start 1\_ and go all the way up to \_0, where the final number's first digit is the digit sum. The second row is therefore:

Sum of the Digits									
	1	2	3	4	5	6	7	8	9
2	1	2	3	4	5	6	7	8	9

## 3-digit numbers

Time to work a bit harder! First of all, let's see if we can write lists of numbers that give the first four results in the table. Have a go yourself before reading on.

Here we work systematically like we did in part c by first considering the digit combinations that give the specific sum.

Digit Sum	1	2	3	4
	<div>100   100</div> <div>[1 solution]</div>	<div>200   200</div> <div>110   110, 101</div> <div>[3 solutions]</div>	<div>300   300</div> <div>210   210, 201, 120, 102</div> <div>111   111</div> <div>[6 solutions]</div>	<div>400   400</div> <div>310   310, 301, 130, 103</div> <div>220   220, 202</div> <div>211   211, 121, 112</div> <div>[10 solutions]</div>

We now have a partially completed row.

Sum of the Digits									
	1	2	3	4	5	6	7	8	9
3	1	3	6	10					

Can you see any patterns here? Have you seen these numbers before? How many 3-digit numbers do you think will have a digit sum of 5?

You may recognise the numbers in the above table as the Triangle numbers from Problem 5. Each new number is formed by adding the next counting number. 1, (+2) 3, (+3) 6, (+4) 10 etc.

With this in mind, we predict that the row will be completed as follows:

	Sum of the Digits								
	1	2	3	4	5	6	7	8	9
3	1	3	6	10	15	21	28	36	45

This feels “nice” though perhaps you are a little uncomfortable in just assuming that the pattern continues without trying to understand why it is happening. We sure are!!

It turns out that looking at the table “as a whole” is very helpful... look at the highlighted cells and see if you can spot something interesting.

		Sum of the Digits								
		1	2	3	4	5	6	7	8	9
Number of Digits	1	1	1	1	1	1	1	1	1	1
	2	1	2	3	4	5	6	7	8	9
	3	1	3	6	10	15	21	28	36	45
	4									
	5									
	6									
	7									
	8									
	9									
	...									

Did you see how once we fill the ones in for the first row and column, each of our entries so far can be found by adding the entry above and the entry to the left? If we can explain why this happens, we are very close to cracking open this whole problem!

It can be helpful in times like this to focus your attention on a specific case. Let’s look at the orange cells. Have a go at seeing if you can explain a way for the six 3-digit numbers with a sum of 3 and the 4 2-digit numbers with a sum of 4 could be used to create the ten 3-digit numbers with a sum of 4. That is... how could,

[300 210 201 111 120 102] + [13 22 31 40] transform into [400 310 301 211 220 202 121 130 112 103]??

Well, the first set of numbers needs to have an extra 1 somewhere to create a digit sum of 4. We could add this 1 to the hundreds, the tens or the ones. Let’s see what this looks like:

[300 210 201 111 120 102] + 100, [300 210 201 111 120 102] + 10, [300 210 201 111 120 102] + 1  
→[400 310 301 211 220 202], [310 220 211 121 130 112], [301 211 202 112 121 103]

If we are lucky one of the above three sets will be a key contribution, however there is always the chance that it is some combination of these.

Let’s remain optimistic for now and look back at the other set, [13 22 31 40], to see if this can fill in the gaps in some way. Now, to adjust this set we need to just make the numbers longer without contributing to the digit sum. The only way to do this is to add a zero somewhere. Our hope is that we can add the zero in a consistent way. We could add the zero to the start the middle or the end...

Start: [13 22 31 40] → [013 022 031 040]

Middle: [13 22 31 40]  $\rightarrow$  [103 202 301 400]

End: [13 22 31 40]  $\rightarrow$  [130 220 310 400]

Which of these options do you think will be more helpful? Why?

We think putting the zero at the start is quite silly, these aren't even numbers!

We could put the zero in the middle, however doing this creates 3 numbers that are in the "add-1-to-the-final-digit-list" and 1 number that is in the "add-1-to-the-first-digit-list." This makes things messy.

Note however, if we add a zero to the final digit of every number then we create 4 numbers that are definitely not part of the final "add-1-to-the-final-digit-list." This is great!!

Let's recap our findings...

We can create a set of 3-digit numbers with a digit sum of 4 by adding 1 to the final digit of all 3-digit numbers with a digit sum of 3 and a zero to the end of all 2-digit numbers with a digit sum of 4.

*In general, we can create a set of  $d$ -digit numbers with a digit sum of  $s$  by adding 1 to the final digit of all  $d$ -digit numbers with a digit sum of  $s-1$  and a zero to the end of all  $(d-1)$ -digit numbers with a digit sum of  $s$ .*

This is great as we have found a way of creating the new sets recursively from the ones before it, though we need to think carefully about why it is impossible to find any other numbers that match our conditions that aren't created in the way described above.

Let's say we are trying to find all  $d$ -digit numbers with a digit sum of  $s$ . Let's also separate the  $d$ -digit number with a digit sum of  $s$  into two groups; those that end with zero and those that don't.

#### Set 1: Numbers that end in a zero

It's quite easy to see that all the numbers that end in a zero have been considered, as we knew we had all possibilities for the  $(d-1)$ -digit numbers and have added a zero to these. There are no other numbers that could end in a zero.

#### Set 2: Numbers that don't end in a zero

As for the remaining numbers, clearly, they can only be formed by adding a 1 to one of the  $d$  digits in the numbers that have a digit sum of  $s-1$ . The challenge that we've already seen is that we could add 1 to any of the  $d$ -digits!

Let's start by adding 1 to the final digit of each number. Each of these numbers will then be a  $d$ -digit number with a digit sum of  $s$ . Great! We just need to convince ourselves that if we were to add a 1 to any other digit we would duplicate a number formed so far (in either Set 1 or Set 2).

For any number ending in a zero, we must add 1 to the final digit (if we don't then the number will already belong in Set 1).

Let  $A$  be the set of all  $d$ -digit numbers where the digit sum is  $s-1$  and the final digit is not zero. One such number in this set is  $a_d a_{d-1} \dots a_2 a_1$  where  $a_d + a_{d-1} + \dots + a_2 + a_1 = s-1$ . Let  $A'$  be the set of all  $d$ -digit number where the digit sum is  $s$  and the final digit of every number in  $A'$  is 1 bigger than the final digit of every number in  $A$ .

We want to prove that it is impossible to create a  $d$ -digit number with a digit sum of  $s$  that isn't part of  $A'$ . Suppose we created a number by adding 1 to  $a_2$ , the second last digit of  $a_d a_{d-1} \dots a_2 a_1$ . Well, this number can also be formed by taking  $a_d a_{d-1} \dots (a_2+1)(a_1-1)$  and adding 1 to its final digit. We know  $a_{d-1} \dots (a_2+1)(a_1-1)$  is part of the Set A as its digit sum is the same as  $a_d a_{d-1} \dots a_2 a_1$  since  $a_d + a_{d-1} + \dots + a_2 + a_1 = a_d + a_{d-1} + \dots + a_2 + 1 + a_1 - 1$ . In fact, every number that is made by taking 1 off the final digit and adding it elsewhere is also part of Set A. For this reason, we don't need to consider adding ones to any other digit and we can be confident that no other  $d$ -digit number with a digit sum of  $s$  and a non-zero final digit are all in  $A'$ .

We are now convinced that the strategy below works. That is

*We can create a complete set of  $d$ -digit numbers with a digit sum of  $s$  by adding 1 to the final digit of all  $d$ -digit numbers with a digit sum of  $s-1$  and a zero to the end of all  $(d-1)$ -digit numbers with a digit sum of  $s$ .*

Using this recursion, we have completed row 3 and 4, have a go at filling in rows 5 and 6 yourself. Note that we have already included the 1s in the first column since there is only one 5-digit number with a digit sum of 1; 10 000.

		Sum of the Digits								
		1	2	3	4	5	6	7	8	9
Number of Digits	1	1	1	1	1	1	1	1	1	1
	2	1	2	3	4	5	6	7	8	9
	3	1	3	6	10	15	21	28	36	45
	4	1	4	10	20	35	56	84	120	165
	5	1								
	6	1								
	7									
	8									
	9									
	...									

Suppose we denote the number of  $d$ -digit numbers with a digit sum of  $s$  as  ${}^dN_s$  where  $s < 10$  we have just shown that:

$${}^dN_s = {}^{d-1}N_s + {}^dN_{s-1}$$

Woo hoo!!! We now have a recursive approach for working out the number of  $d$ -digit numbers with a digit sum of  $s$  (where  $s < 10$ ).

Now recall that the question asked us "How many numbers?" not just "How do you work out how many numbers?" Let's see if we can complete this extra challenge.



We’ve now filled in a bit more of our table. Using the recursion, fill in the yellow and orange diagonals.

		Sum of the Digits								
		1	2	3	4	5	6	7	8	9
Number of Digits	1	1	1	1	1	1	1	1	1	
	2	1	2	3	4	5	6	7		
	3	1	3	6	10	15	21			
	4	1	4	10	20	35				
	5	1	5	15	35					
	6	1	6	21						
	7	1	7							
	8	1								
	9									
	10									
	...									

Now interestingly, if you turn your head 45 degrees, these are the exact same numbers that are found in Pascal’s triangle... Can you see how the numbers in each new row of Pascal’s triangle are formed from the row above it? What values do you think will be in the next row?

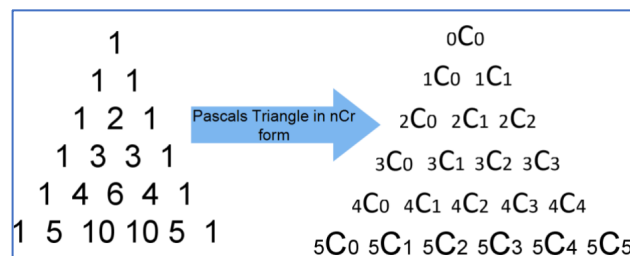
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Excitingly there is a formula for the values in Pascal's triangle. The formula is quite complex, though included below for those interested. Note that the formula uses a symbol that we haven't yet come across, the factorial symbol  $!$ . This symbol is best explained using examples:

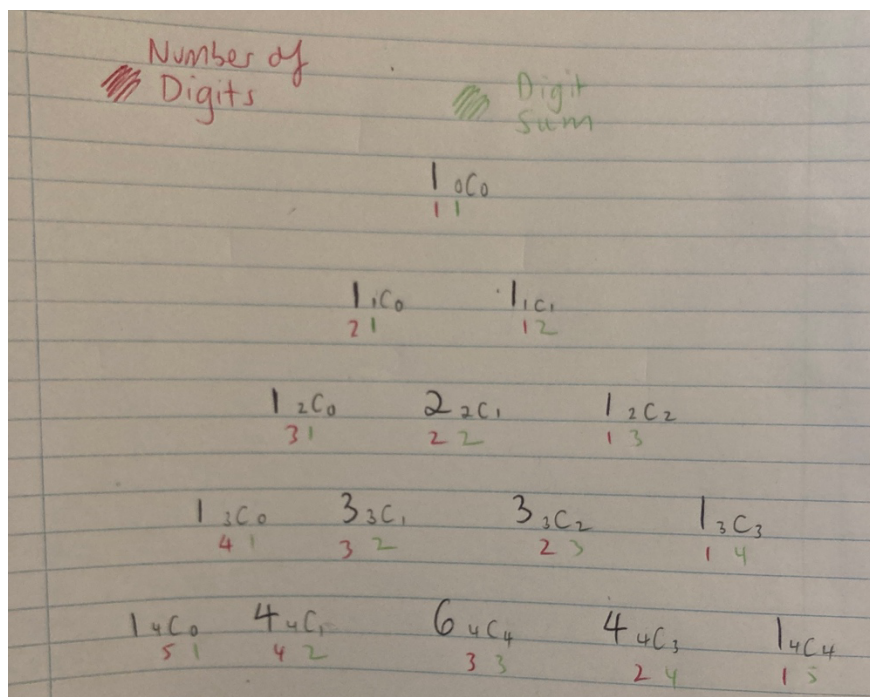
$$5! = 5 \times 4 \times 3 \times 2 \times 1$$

The numbers in Pascal's triangle are calculated using the formula below where  $n$  and  $r$  closely relate to the row and row position of each value.

$${}_nC_r = \frac{n!}{r!(n-r)!}$$



The following picture shows how our results are contained in Pascal's triangle.



What do you notice about the red and green values in each row? Could you come up with a formula for the number of d-digit numbers with a sum of s (where  $k < 10$ )? Look closely at the values either side of the C (the n and r in  $nCr$ ) and how they connect to the numbers in red and green.

We noticed that red+green-2 = n and green-1 = r. With red = the number of digits (d) and green equal the sum (s) and how we denoted the number of d-digit numbers with a digit sum of s as  ${}^dN_s$  where  $s < 10$  we therefore have:

$$\begin{aligned} {}^dN_s &= {}_{d+s-2}C_{s-1} = \frac{(d+s-2)!}{(s-1)!(d+s-2-[s-1])!} \\ &= \frac{(d+s-2)!}{(s-1)!(d-1)!} \end{aligned}$$

Let's check if our formula works. We know from our work above that there are 10 3-digit numbers with a digit sum of 4.

$${}^dN_s = {}^3N_4 = \frac{(3+4-2)!}{(4-1)!(3-1)!} = \frac{5!}{3!2!} = \frac{5!}{3!} \div 2 = \frac{5 \times 4}{2} = 10$$

And why this formula works, will have to be a job for another day!

## Problem 9

- a. What is the sum of all underlined digits, that is the ones digits, the units digits, of 10, 11, 12, ..., 19?

We have  $0+1+2+3+\dots+9$ , ah our Triangle numbers. The answer is 45.

You may have calculated this by computing  $9 \times 10 / 2$  or perhaps you did  $(1+9)+(2+8)+(3+7)+(4+6) + 5 = 4 \times 10 + 5$ .

Does it make sense that the sum is odd? Why?

- b. What is the sum of all the ones digits of 20, 21, 22, ..., 29?

What a kind question we hear you say. Yes, this is the same sum as before.  $0+1+2+3+\dots+9=45$

- c. What is the sum of all the ones digits of all 2-digit numbers, 10, 11, 12, ..., 99?

How many copies of the original sum do we have now? Well, we see nine copies of the sum  $0+1+2+\dots+9$  so we calculate it efficiently  $9 \times 45 = 405$ .

- d. What is the sum of all the tens digits of the 2-digit numbers?

We have 10 1s, 10 2s, 10 3s, all the way up to 10 9s. What is this sum?

Do you get  $10 \times (1+2+3+\dots+9) = 450$ ? We do.

- e. Now, what is the sum of the digits of all 2-digit numbers, 10 to 99?

How can the previous answers help us?

Let's add our answers from parts c and d together. We get  $405+450=855$

- f. What is the sum of the digits of all 3-digit numbers, 100 to 999?

There are 900 numbers, and so 900 ones digits, 900 tens digits and 900 hundreds digits. 1-tenth of the ones digits are 0, 1-tenth of the ones digits are 1 and so on so there are 90 of each digit in a ones place. The same can be said for the tens digits too. However, the hundreds digits cannot be 0 so 1-ninth of the hundreds digits are 1, 1-ninth of the hundreds digits are 2 and so on, so there are 100 of each digit in a hundreds place.

Can you use these ideas to work out the sum of the digits in all 3-digit numbers?  
Here's what we did:

The sum of the digits in all 3-digit numbers

= sum of all ones digits + sum of all tens digits + sum of all hundreds digits

$= 90 \times (0+1+2+\dots+9) + 90 \times (0+1+2+\dots+9) + 100 \times (1+2+\dots+9)$

$= 90 \times 45 + 90 \times 45 + 100 \times 45$

$= 12600$

**Further Exploration:** Wendy tried to find the formula for the sum of the digits of all  $n$  digit numbers, this is what she got:  $45 \times 10^n + 9 \times n \times 45 \times 10^{n-1}$ . Is she right? Either convince us that she is correct, or fix the formula and convince us why it works.

There are 90 2-digit numbers, 900 3-digit numbers and 9000 4-digit numbers. How many  $n$ -digit numbers are there?

Continuing the pattern, you might see that there are  $9 \times 10^{n-1}$   $n$ -digit numbers. Using the ideas from the 3-digit numbers you might like to start thinking about the sum of the ones, tens, hundreds etc. Just save the  $n^{\text{th}}$  place value position until last as it is a little different.

Now we have  $9 \times 10^{n-1}$  numbers, which means there are  $9 \times 10^{n-1}$  digits in each of the  $n$  place value positions. For the ones digit up to the  $(n-1)^{\text{th}}$  place value position the digits can be 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. So 1-tenth of each place value position will be each digit. One-tenth of  $9 \times 10^{n-1}$  is  $9 \times 10^{n-2}$  and so the sum of the digits in the first  $n-1$  place value positions is:

$$(n-1) \times (0+1+2+\dots+9) \times 9 \times 10^{n-2} = (n-1) \times 45 \times 9 \times 10^{n-2}$$

The digits in the first place value position can only be 1, 2, 3, 4, 5, 6, 7, 8 or 9. So 1-ninth of the  $9 \times 10^{n-1}$   $n^{\text{th}}$  place value position will be each of these digits. One-ninth of  $9 \times 10^{n-1}$  is  $10^{n-1}$  and so the sum of the digits in the  $n^{\text{th}}$  place value position is:

$$(1+2+\dots+9) \times 10^{n-1} = 45 \times 10^{n-1}$$

Wendy was close, though the general case is given by:

$$45 \times 10^{n-1} + (n-1) \times 45 \times 9 \times 10^{n-2}$$

Let's check that this works for the sum of the digits in the 3-digit numbers. Letting  $n=3$  we get:

$$45 \times 10^2 + 2 \times 45 \times 9 \times 10 = 4500 + 8100 = 12600$$

Awesome!!

## Problem 10

Below shows the different ways we can make a line out 5¢ and/or 10¢ coins that totals 20¢.



- a. Lay out the 5¢ and 10¢ coins that make up 30¢ in a line. How many orders are there for these coins?

This problem has a little bit of a Leo the Rabbit feel to it... Let's approach it in a similar way. We'll start with the 10¢ coin(s) at the start of a pile. If there is more than one 10¢ coin, we move the final one up at first before moving any of the earlier ones.

It might get a little fiddly to write 10 and 5 so let's let  $a = 5¢$  coin and  $b = 10¢$  coin.

Zero x 10¢ coins	One x 10¢ coin	Two x 10¢ coins	Three x 10¢ coins
aaaaaa	baaaa, abaaa, aabaa, aaaba, aaaab	bbaa, baba, baab, abba, abab aab	bbb

There are **13 ways** of arranging the 5 and 10 cent coins to make 30¢.

- b. Lay out the 5¢ and 10¢ coins that make up 40¢ in a line. How many orders are there for these coins?

The table will have the following headings, can you fill them in?

Our answer is on the next page.

Zero x 10¢ coins	One x 10¢ coin	Two x 10¢ coins	Three x 10¢ coins	Four x 10¢ coins

Zero x 10¢ coins	One x 10¢ coin	Two x 10¢ coins	Three x 10¢ coins	Four x 10¢ coins
aaaaaaaa	baaaaaa, abaaaaa, aabaaaa, aaabaaa, aaaabaa, aaaaaba, aaaaaab	bbaaaa, babaaa, baabaa, baaaba, baaaab  abbaaa ababaa, abaaba, abaaab  aabbba, aababa, aabaab  aaabba, aaabab  aaaabb	bbbaa, bbaba, bbaab babba, babab baabb  abbba, abbab ababb  aabbb	bbbb

There are **34 ways** of arranging the 5 and 10 cent coins to make 40¢.

- c. Lay out the 5¢ and 10¢ coins that make up 60¢ in a line. How many orders are there for these coins?

Okay, so the numbers we have found so far (5, 13 and 34) came up in the Leo problem, though it's definitely not the whole picture as with Leo we had 5, 8, 13, 21 and 34. Why is this??

It might have something to do with only considering the multiples of 10, when we can also create totals like 25¢ and 35¢ too.

We can make this problem even more "Leo-like" by considering all the cases and seeing if we can justify another recursive relationship (where new solutions are found by doing something with the previous ones).

Here's what we know so far:

Amount of money	5¢	10¢	15¢	20¢	25¢	30¢	35¢	40¢
Number of ways				5		13		34

We can quickly work out the number of ways of making 5¢, 10¢ and 15¢.

- 5¢ - one way (5¢)
- 10¢ - two ways (5+5, 10)
- 15¢ - three ways (5+5+5, 10+5, 5+10)

So now we have:

Amount of money	5¢	10¢	15¢	20¢	25¢	30¢	35¢	40¢
Number of ways	1	2	3	5		13		34

And a Fibonacci sequence is appearing again. This is 100% a Leo-type problem!

From 15¢ onwards we see that the sum of the two previous solutions gives us our next one. This happens because we can create a total of  $(n+1)$  lots of 5¢ by putting 5¢ at the start of all solutions that total  $n$  lots of 5¢ and putting 10¢ at the start of all solutions that total  $(n-1)$  lots of 5¢.

We can now work our way up to how many ways we can create 60¢.

Amount of money	5¢	10¢	15¢	20¢	25¢	30¢	35¢	40¢	45¢	50¢	55¢	60¢
Number of ways	1	2	3	5	8	13	21	34	55	89	144	233

We see that there are **233 ways** of arranging the 5 and 10 cent coins to make 60¢.



**Further exploration:** Consider what happens when we have 5¢, 10¢ and 20¢ coins.

This is Leo-ish, but we need to be careful. In Leo we went from having 1- and 2-hops to 1-, 2- and 3-hops. The equivalent problem here would be going from 5¢ and 10¢ coins to 5¢, 10¢ and 15¢ coins. However, here we have 5¢, 10¢ and 20¢ coins. How does this change things?

Well, we just need to add different cases.

The problem starts off the same, however for 20¢ we have one extra solution.

Amount of money	5¢	10¢	15¢	20¢	25¢	30¢	35¢	40¢	45¢	50¢	55¢	60¢
Case Number	1	2	3	4	5	6	7	8	9	10	11	12
Number of ways	1	2	3	5	6							

For ease of discussion let's add another row to our table. "The case number" which is effectively the how many 5¢ make up the amount of money.

For Case Number 5, 25¢, we can add a 5¢ coin to the start of Case Number 4, or we can add a 10¢ coin to the start of the solution for Case Number 3 or we can add a 20¢ coin to the start of the solution for Case Number 1. So, we get  $1+2+6=9$  ways.

Have a go at completing the table above and see if you can come up with a way of using previous solutions to get new ones. Think carefully about which previous solutions are required.

The following page contains our full solution, though we are sure that you'll be most satisfied if you have a go on your own first.

The approach for making 25¢ by adding 5¢ to the 20¢, adding 10¢ to the 15¢ and 20¢ to the 5¢ generalises to all cases. That is, to work out the number of ways for Case Number  $n+1$  we add 5¢ to the solutions for Case Number  $n$ , we add 10¢ to the solutions for Case Number  $n-1$  and add 20¢ to the solutions for Case Number  $n-3$ .

We now complete our table and see that there are 520 ways of creating 60¢ using 5¢, 10¢ and 20¢ coins.

Amount of money	5¢	10¢	15¢	20¢	25¢	30¢	35¢	40¢
Case Number	1	2	3	4	5	6	7	8
Number of ways	1	2	3	6	10 (6+3+1)	18 (10+6+2)	31 (18+10+3)	55 (31+18+6)

Amount of money	45¢	50¢	55¢	60¢
Case Number	9	10	11	12
Number of ways	96 (55+31+10)	169 (90+55+18)	296 (169+96+31)	520 (296+169+55)

## Problem 11

In the following questions, we consider a  $4 \times 1$  rectangle and a  $1 \times 4$  rectangle as the same rectangle. Whereas a  $2 \times 2$  rectangle and a  $1 \times 4$  rectangle are different rectangles (yes, a square is a rectangle!). The same approach is taken for rectangular prisms and so a  $1 \times 1 \times 4$ , a  $1 \times 4 \times 1$  and a  $4 \times 1 \times 1$  are all the same rectangular prism too.

a. How many different rectangles of area  $25\text{m}^2$  can be made using  $1\text{cm}^2$  tiles? List them all.

The key ideas here are that a rectangle has area length by width and the product of length by width here is 25. Now the only factors of 25 are 1, 5, and 25. Let's look at these systematically.

If length is 1cm then width is 25 cm to give us a  $1 \times 25$  rectangle.

If length is 5 cm, then width is 5 cm and we get a  $5 \times 5$  rectangle.

If length is 25 cm, then width is 1 cm, so here we have a  $25 \times 1$  rectangle.

But a  $25 \times 1$  rectangle is the same as a  $1 \times 25$  rectangle. This means that we can only have two rectangles of area  $25\text{cm}^2$ . These are  $1 \times 25$  and  $5 \times 5$ .

b. How many different rectangles of area  $64\text{cm}^2$  can be made using  $1\text{cm}^2$  tiles? List them all.

Here it's the factors of 64 that are important. These are 1, 2, 4, 8, 16, 32 and 64. We again work systematically.

Factor 1 gives  $1 \times 64$ ; factor 2 gives  $2 \times 32$ ; factor 4 gives  $4 \times 16$ ; factor 8 gives  $8 \times 8$ ; factor 16 gives  $16 \times 4$ ; factor 32 gives  $32 \times 2$ ; factor 64 gives  $64 \times 1$ .

The different rectangles are therefore  $1 \times 64$  ( $= 64 \times 1$ ),  $2 \times 32$  ( $= 32 \times 2$ );  $4 \times 16$  ( $= 16 \times 4$ ); and  $8 \times 8$ . So, there are four rectangles in this case.

c. How many different rectangular prisms of volume  $20\text{m}^3$  can you make using  $1\text{cm}^3$  blocks? List them all.

Here we need to find three factors of 20. The method is essentially the same as the rectangles but with three numbers multiplying together to get 20. The different factors of 20 are 1, 2, 4, 5, 10, 20.

To ensure we count all possibilities we work systematically (of course!). We first look at all rectangular prisms for which the smallest side length is 1. These are easily created using the factor pairs for two-dimensions of our rectangular prism and making 1 the other dimension. Here we have:  $1 \times 1 \times 20$ , then  $1 \times 2 \times 10$  and  $1 \times 4 \times 5$ .

We then consider the rectangular prisms with a smallest side of 2. If one dimension is 2 then the other two dimensions must have a product of 10. The factors of 10 are 1 and 10, and 2 and 5. We can't use the 1 and 10 as then the smallest side would be 1 (and we've already counted these). So we must have one more option being  $2 \times 2 \times 5$ .

Is it possible to have a rectangular prism where the smallest side is larger than 2? Well, the next smallest factor is 4 so we'd need the two other dimensions to have a product of 5. A product of 5 will produce a factor smaller than 4 so this cannot be done.

We are now confident that we have all solutions. The four prisms with a volume of  $20\text{cm}^3$  are:  $1 \times 1 \times 20$ ,  $1 \times 2 \times 10$ ,  $1 \times 4 \times 5$  and  $2 \times 2 \times 5$ .

d. How many different rectangular prisms of volume  $60\text{m}^3$  can you make using  $1\text{cm}^3$  blocks? List them all.

The method here is the same as in part c. The factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60. (If we carefully find all the factors from 1 to 6, why can we just write down the bigger factors?). We proceed using the systematic approach from part c.

Prisms with the smallest side of 1 are:  $1 \times 1 \times 60$ ,  $1 \times 2 \times 30$ ,  $1 \times 3 \times 20$ ,  $1 \times 4 \times 15$ ,  $1 \times 5 \times 12$ ,  $1 \times 6 \times 10$ .

Prisms with the smallest side of 2 are:  $2 \times 2 \times 15$ ,  $2 \times 3 \times 10$ ,  $2 \times 5 \times 6$ .

The prism with the smallest side of 3 is:  $3 \times 4 \times 5$ .

Why can't we have a prism with the smallest side of 4?

So, the answer to the question is 10 different prisms.

Further exploration: A *plong* is made up of  $a$  blocks in the first dimension,  $b$  blocks in the second dimension,  $c$  blocks in the third dimension and  $d$  blocks in the fourth. The size of a *plong* is calculated by multiplying the dimensions:  $a \times b \times c \times d$ . How many different *plongs* of size 60 exist? What about even higher dimensions?

Recall that the factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60. Using these systematically to produce 60 in 4-dimensions we get:

$1 \times 1 \times 1 \times 60$ ,

$1 \times 1 \times 2 \times 30$ ,  $1 \times 1 \times 3 \times 20$ ,  $1 \times 1 \times 4 \times 15$ ,  $1 \times 1 \times 5 \times 12$ ,  $1 \times 1 \times 6 \times 10$  (note that 2, 3, 4, 5, 6 are the factors less than  $\sqrt{60}$ ; why is this important?)

$1 \times 2 \times 2 \times 15$ ,  $1 \times 2 \times 3 \times 10$ ,  $1 \times 2 \times 5 \times 6$ ,  $1 \times 2 \times 6 \times 10$ ,  $1 \times 3 \times 4 \times 5$

$2 \times 2 \times 3 \times 5$  (is 2 the only factor that can be repeated other than 1?)

Why can't we have four different factors of 60 (not using 1) whose product is 60?

What was the systematic method used here?

We'll leave it to you to explore higher dimensions.

## Problem 12

a. What is the final digit of  $2^5$ ?

For a start this is all about multiplying 2 by itself. So let's do that.

$2^1 = 2$ ,  $2^2 = 4$ ,  $2^3 = 8$ ,  $2^4 = 16$ ,  $2^5 = 32$ . Hence, the final digit is 2.

b. What is the final digit of  $2^{20}$ ?

There is nothing stopping you now finding all of the powers of 2 up to 20. To save time you might write a computer program to do this, especially if you want to do parts c and d the same way. But maths can be prettier than that.

From part a, note that the last digits of  $2^1$ ,  $2^2$ ,  $2^3$ ,  $2^4$ , and  $2^5$  are 2, 4, 8, 6 and 2, respectively. Now see what happens for  $2^6$ ,  $2^7$ ,  $2^8$ ,  $2^9$ , and  $2^{10}$ . Do the end digits of these numbers go 4, 8, 6, 2, 4. Try that again with  $2^{11}$ ,  $2^{12}$ ,  $2^{13}$ ,  $2^{14}$ ,  $2^{15}$ ,  $2^{16}$ ,  $2^{17}$ ,  $2^{18}$ ,  $2^{19}$ , and  $2^{20}$ . This should give you 6 for the answer to this part.

c. What is the final digit of  $2^{2023}$ ?

The question is, do you really want to find all of the powers of 2 up to 2023? If you think about what we learnt in part b you will likely find a quicker way as there is a lovely pattern going on. It looks like 2, 4, 8, 6, followed by 2, 4, 8, 6, ... followed by ... which means it all depends on four possible final digits.

Final Digit	2	4	8	6
Power of 2	$2^1, 2^5, 2^9, 2^{13}$	$2^2, 2^6, 2^{10}, 2^{14}$	$2^3, 2^7, 2^{11}, 2^{15}$	$2^4, 2^8, 2^{12}, 2^{16}$

Can you see that patterns in the powers of 2 that have a consistent final digit? Because it cycles every 4, all powers in a set are a multiple of 4 plus some fixed amount. For example, every number that ends in a 6 is 2 to the power of a multiple of 4, whereas every number that ends in a 2 is 2 to the power of 4 plus 1. Since  $2023 = 4 \times 505 + 3$ ,  $2^{2023}$  must fit in the group of 2 to the power of a multiple of 4 plus 3 and so, the answer we want here is 8.

d. What is the final digit of  $3^{2023}$ ?

The question is, does 3 behave like 2 in part c?  $3^1 = 3$ ,  $3^2 = 9$ ,  $3^3 = 27$ ,  $3^4 = 81$ ,  $3^5 = 243$ ,  $3^6 = 729$ , and so on. Since  $2023 = 4 \times 505 + 3$ . Does that make you think that maybe the answer we want here is 7?

**Further Exploration:** Do the powers of other numbers have interesting patterns?

Using excel we can quickly produce a table of the final digit of different powers of numbers.

Value of x	1	2	3	4	5	6	7	8	9	10
x to the power of 1	1	2	3	4	5	6	7	8	9	0
x to the power of 2	1	4	9	6	5	6	9	4	1	0
x to the power of 3	1	8	7	4	5	6	3	2	9	0
x to the power of 4	1	6	1	6	5	6	1	6	1	0
x to the power of 5	1	2	3	4	5	6	7	8	9	0
x to the power of 6	1	4	9	6	5	6	9	4	1	0
x to the power of 7	1	8	7	4	5	6	3	2	9	0
x to the power of 8	1	6	1	6	5	6	1	6	1	0
x to the power of 9	1	2	3	4	5	6	7	8	9	0
Last digit repeats every...	1	4	4	2	1	1	4	4	2	1

Oh and does there always exist a pattern, and the numbers repeat quite quickly. The maximum number of digits before a repeat is 4!

When is the first time the final two digits of a power of 2 repeat? What about the final two digits of a power of 3? Or 4? Etc.

That looks like a lot of work. Maybe it's computer time again. Or is there a neat way to do these questions too?

## Problem 13

There are some cubes on the table. Alice and Blair alternatively remove one or two cubes. The winner is the one who takes the last cube. On the principle of 'ladies first', Alice always takes the first turn. Alice and Blaire are expert players, and depending on the number of cubes they each know who will win.

a. Who wins when there are one cube?

Well Alice has to take the first turn. So she takes the only cube. Blaire has no chance to take any cube, so Alice took the last cube and so Alice won.

b. Who wins when there are two cubes.

Alice takes them both, Alice wins.

c. Who wins when there are three cubes?

What should Alice do? She has the first turn and she can take one or two cubes. If she takes one cube, Blaire will take two and win. If she takes two cubes, Blaire will take one and win.

Now Alice has no other options. She's not allowed to take no cubes or three cubes. So Blaire must win.

d. Who wins when there are four cubes?

Alice can take one or two cubes. If she takes one cube there are three left. What can Blaire do? If he takes one, then Alice takes two and wins; if he takes two, then Alice takes two and wins. In both cases she wins. We know this because we have covered all of the possibilities.

Another way to look at this is that if Alice can reduce the number of cubes to three, then Blaire will be in the same position as she was at the start of part b. So Alice can put Blaire in a bad position by taking just one cube on her first move.

But what if Alice takes two cubes? Well, Blaire will take the remaining two and win. However, Alice is going to play as well as she can So she knows that one is better than two, and so takes just one cube to start with, and wins.

e. Who wins when there are 21 cubes?

Now it's not easy to solve this if there are 21 cubes on the table and Alice and Blaire madly pick cubes at random. What we suggest is that you move gradually and now see what happens with four, five, six cubes and so on. This might lead to a method of working with 21 cubes.

**Five cubes.** Having learnt from four cubes, Alice realises that if she takes two cubes on her first turn, she'll have left Blaire with three cubes and so he will lose (if she plays carefully on her second turn). How will the game play out in Alice's favour?

**Six cubes.** If Alice takes one cube, then Blaire will now be in the same position as she was in the five-cube situation. So she'll lose. Look at the five-cube argument but interchange Alice and Blaire.

If Alice takes two cubes at her first turn, Blaire will be facing four cubes. This is exactly the four-cube situation in part c, but Blaire is now the first player and he'll win!

**Seven cubes.** Alice can take one cube and reduce the number of cubes to six. This is great, since Alice couldn't win with six cubes, Blaire won't be able to win either!

But Alice shouldn't take two cubes because then Blaire will have five to think about and we know from above that if you have five cubes you can make sure you win. Alice should definitely take one cube.

**Eight cubes.** Alice can guarantee a win again by taking two cubes and forcing Blaire into the losing situation of having six cubes.

**Think.** Now Blaire wins when there are three and six cubes. What about nine cubes? With nine cubes, no matter what Alice does, on his first move he can reduce the number of cubes to six and we know that Alice will lose that game.

And the same thing will happen with 12 cubes. Whatever Alice does, Blaire can reduce the pile to nine cubes and win.

The same arguments hold in favour of Blaire for 15, 18 and 21 cubes. So, Alice must lose the 21-cube game.

f. Who wins when there are 31 cubes?

Alice and Blaire now know that Blaire will win if the number of cubes is a multiple of three. She also knows that the first player to start on a multiple of three cubes will lose. So Alice takes one cube on her turn and Blaire loses the 30-cube situation. Alice wins on 31.

The same thing would be true for 32 cubes. Indeed, Alice now knows how to win for any number of cubes that isn't a multiple of 3.

g. How do Alice and Blaire know who will win?

If there is a multiple of three cubes Blaire will win, well, unless he falls asleep. If the number of cubes isn't a multiple of three, Alice should win if she has learnt anything we've tried to show her above.

**Further Exploration:** Blocks are placed arbitrarily in two piles. When it's their turn Alice and Blair can take either one or two blocks, provided they come from one pile. The winner is the person with the last block. Again, Alice goes first. Under what circumstances will Alice win? What if there were more piles?

If the two piles have the same number, then Blaire will win. He just does to the second pile what Alice did to the first pile.

Will Alice win otherwise? How good a strategy is it for Blaire to use the same strategy as in the equal piles? What happens if the first pile gets reduced completely? Is that good for Alice or Blaire? Does the difference between the original piles help?

Having thought all of these thoughts and others, you may not have a really good idea to tackle the problem. So, it's not a bad idea about now to look at a specific example. How about one pile with 5 cubes and the other with 4? Alice might try taking one cube from the 5-pile. Ah, but then she could follow Blaire and do what he does, but on the other pile.

She would tackle a 6- and a 4-pile in a similar way. But what about a 7- and a 4-pile? Should she take cubes away from the smaller pile first?

Note that experimenting with particular examples is always a good thing to try if you are stuck for an idea.



To work through this problem in a systematic way we started by setting up a table showing all simple cases writing down the winner. Notice that we can assume that the first pile is equal to or larger than the second pile as it doesn't actually matter which stack is pile one and which stack is pile two.

We started working along the top row – highlighted in blue, filling in what we know from the original problem.

We can go quite a bit further knowing that Blaire is in a winning position if both piles have the same number of cubes – highlighted in yellow (recall the copying strategy mentioned above). Now if the larger pile is one or two bigger than the other – highlighted in green Alice can win (as she can take one or two cubes and force Blaire into the losing position of there being equal sized piles).

Focusing our attention on the second row, we consider the case where there are four cubes in the first pile and one cube in the second. Now, if Alice takes one or two cubes from the larger pile she leaves Blaire in a winning position (her winning position from piles of 3 and 1 or piles 2 and 1). Similarly, if Alice takes the lone cube from the second pile, she leaves Blaire in the winning position of a pile of 4.

Now if there were 5 and 1 then all is good, Alice takes 1 pile from the large pile and forces Blaire into the losing position. Similarly for 6 and 1. The pattern continues along. For Alice, she will lose if the piles are equal or if the difference between the piles is a multiple of three. The good news for Alice is that she'll win two out of three games. Also, Alice is pretty knowledgeable now so chances are she'll be able to beat most people now regardless of the number of cubes... except you and Blaire of course!

		Number of cubes in the first pile								
		1	2	3	4	5	6	7	8	9
Number of cubes in the second pile	0	A	A	B	A	A	B	A	A	B
	1	B	A	A	B	A	A	B	A	A
	2	x	B	A	A	B	A	A	B	A
	3	x	x	B	A	A	B	A	A	B
	4	x	x	x	B					
	5	x	x	x	x	B				
	6									
	7									
	8									
	9									

Diagram 1. Winner of Alice and Blaire's Game with two piles

The idea of there being more piles appears a little overwhelming but it can be approached in much the same way. We just need to draw a 3D table... hmmm since this isn't really practical on a piece of paper, we can just create different tables for different sizes of the third pile.

Again, we'll assume that the size of the piles is  $\text{Pile One} \geq \text{Pile Two} \geq \text{Pile Three}$ . If we omit the first row. Diagram 1 shows us the case where there are zero cubes in the third pile.

What if there was 1 cube in the third pile?

The 1, 1, 1 situation is great for Alice as she takes one cube and leaves Blaire with two identical piles. Similarly for the 2, 1, 1 – Alice simply takes 2 cubes from the first pile. What about the 3, 1, 1? Well if she takes from the large pile she leaves Blaire with the winning strategy (a 2, 1, 1 or a 1, 1, 1).

Similarly if she takes from a smaller pile Blaire is left with 3, 1, 0 and we know from Diagram 1 that this is a winning position. Unfortunately, Alice can't win here.

The entire table can now be filled in as every case is just a build up of something we already know.

		Number of cubes in the first pile								
		1	2	3	4	5	6	7	8	9
Number of cubes in the second pile	0	x	x	x	x	x	x	x	x	x
	1	A	A	B	A	A	B	A	A	B
	2	x	A	A	B	A	A	B	A	A
	3	x	x	A	A	B	A	A	B	A
	4	x	x	x	A	A	B	A	A	B
	5	x	x	x	x	A	A	B	A	A
	6									
	7									
	8									
	9									

Diagram 2. Winner of Alice and Blaire's Game with three piles, and 1 cube in the third pile

Let's look at the situation where we have two cubes in the third pile.

The initial situation is where we have 2, 2, 2. Alice starts by removing one of the piles and puts Blaire in a losing position. What about 3, 2, 2? On her turn, Alice can turn this into a 2, 2, 2, or 1, 2, 2, or 3, 2, 1 or 3, 2, 0. Using our previous entries we see that these are all bad for Alice and she will lose. We have our starting point and our patterns continue!

		Number of cubes in the first pile								
		1	2	3	4	5	6	7	8	9
Number of cubes in the second pile	0	x	x	x	x	x	x	x	x	x
	1	x	x	x	x	x	x	x	x	x
	2	x	A	B	A	A	B	A	A	B
	3	x	x	A	B	A	A	B	A	A
	4	x	x	x	A	B	A	A	B	A
	5	x	x	x	A	B	A	A	B	A
	6									
	7									
	8									
	9									

Diagram 2. Winner of Alice and Blaire's Game with three piles, and 2 cube in the third pile

At this point we would predict that Blaire will win if there are 3 cubes in each pile. Let's see! Well Alice can turn this into a 3, 3, 2 or a 3, 3, 1. Both these put Blaire in a winning position.

So what is the overall pattern here? When is Blaire winning? Well the first time Blaire wins on each table is as follows: 3, 0, 0; 3, 1, 1; 3, 2, 2. From here, if any pile increases by a multiple of 3 Blaire wins again. This is tough to generalise, and we've only considered up to 3 piles!! Oh dear!!

While we haven't got to our intended destination, and that can be a bit frustrating, we've done some great exploring and worked through some really challenging ideas. For now, that will have to be sufficient!

Back to you, we recommend you look up Nim on the web. Can you decide who wins there?

## Problem 14

a. How many factors does 2 have?

2: 1 and 2.

b. How many factors does 4 have?

3: 1, 2 and 4.

c. How many factors does 8 have?

4: 1, 2, 4 and 8

d. How many factors will  $2^n$  have?

Okay, so now things get a bit more interesting... we've effectively been asked: "what happens in general?" Well, let's consider our answers to parts a-c.  $2^1$  had 2 factors,  $2^2$  had 3 factors and  $2^3$  had 4 factors. This pattern seems to imply that  $2^n$  will have  $n+1$  factors. Actually, I'm confident that it will as the factors of  $2^n$  are  $1, 2^1, 2^2, \dots, 2^n$ . Excellent!

e. How many factors will  $3^n$  have?

Considering all our work so far, it is likely that  $3^n$ , like  $2^n$ , has  $n+1$  factors. Let's check with a few smaller cases. 3 has 2 factors: 1 and 3.  $3^2$  has 3 factors: 1, 3 and 9.  $3^3$  has 4 factors: 1, 3, 9 and 27. So yes, in general  $3^n$  will have  $n+1$  factors:  $1, 3^1, 3^2, \dots, 3^n$ . Again, excellent!

f. How many factors will  $4^n$  have?

Well this is too simple right!?! Will  $4^n$  have  $n+1$  factors? Let's check a few smaller cases. 4 has 3 factors: 1, 2 and 4...what!?! Things don't even work with the first case! Okay, let's work through the smaller cases carefully.

$4^1 = 4$ : 1, 2, 4 (3 factors)

$4^2 = 16$ : 1, 2, 4, 8, 16 (5 factors)

$4^3 = 64$ : 1, 2, 4, 8, 16, 32, 64 (7 factors)

Okay, so we have a nice pattern, which is great. But what is going on here? Well  $4^1$  is also equal to  $2^2$ , and  $4^2 = 2^4$  and  $4^3 = 2^6$ . Which means that  $4^n = 2^{2n}$  and so that's why  $4^n$  has  $2n+1$  factors.

g. How many factors will  $10^n$  have?

Well, we won't be fooled again here. I wonder what the pattern is for powers of 10? Let's start with some smaller cases.

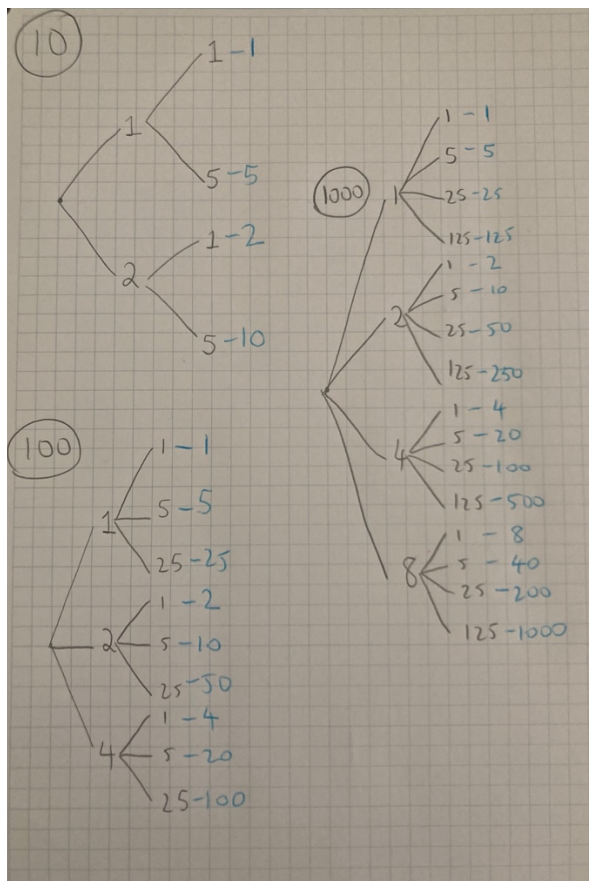
$10^1 = 10$ : 1, 2, 5, 10 (4 factors)

$10^2 = 100$ : 1, 2, 4, 5, 10, 20, 25, 50, 100 (9 factors)

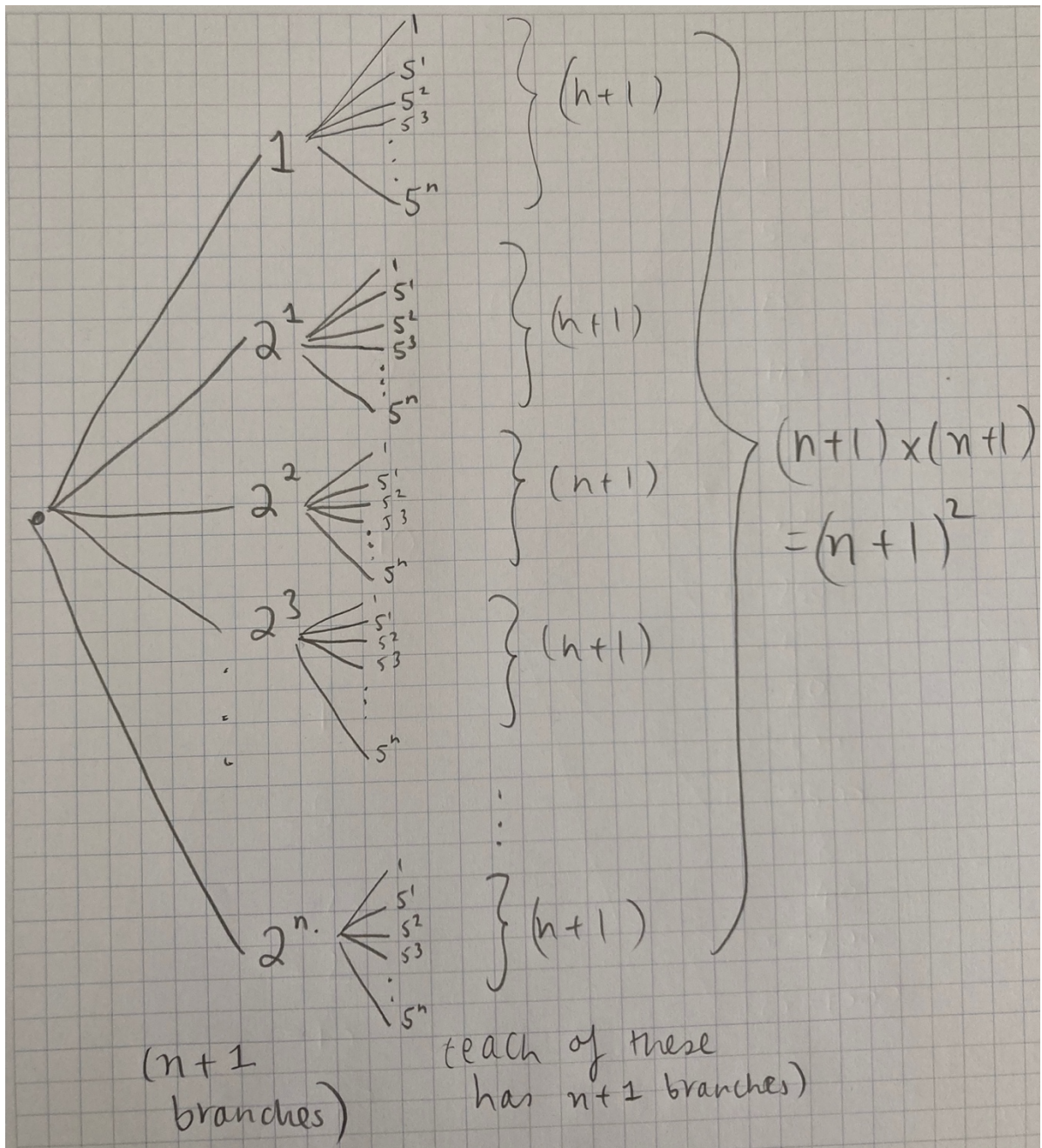
$10^3 = 1000$ : 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 125, 200, 250, 500, 1000 (16 factors)

The number of factors is always a square number. Using the pattern we see here, we predict that  $10^n$  has  $(n+1)^2$  factors. Hmm... but why would this be??

The result relies on the prime factorisation of a number.  $10=2 \times 5$ ,  $10^2=2^2 \times 5^2$  and  $10^3=2^3 \times 5^3$ . It is possible to list all factors of each of these numbers systematically using a tree diagram.



This process can be generalised for  $10^n$  as shown below, confirming that  $10^n$  does indeed have  $(n+1)^2$  factors.



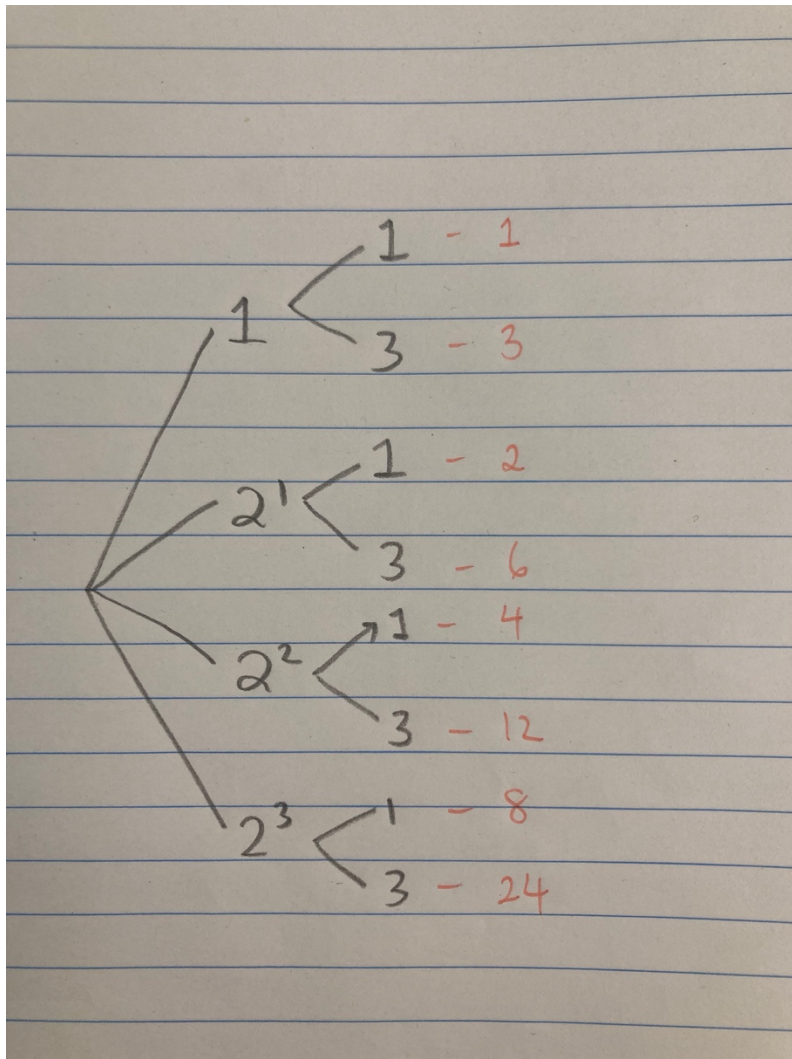
**Further exploration:** The table below shows the product of the numbers in the grey cells, and then in brackets shows how many factors the number has. Complete the table below and use it to explore the number of factors of  $2^n \times 3^m$ .

$\times$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$
$3^0 = 1$	1 (1)	2 (2)	4 (3)	8 (4)
$3^1 = 3$	3 (2)	6 (4)	12 (6)	24 (8)
$3^2 = 9$	9 (3)	18 (6)	36 (9)	72 (12)
$3^3 = 27$	27 (4)	54 (8)	108 (12)	216 (16)

The number of factors  $2^n \times 3^m$  is  $(n+1)(m+1)$ . This result is closely related to the powers of 10 in the



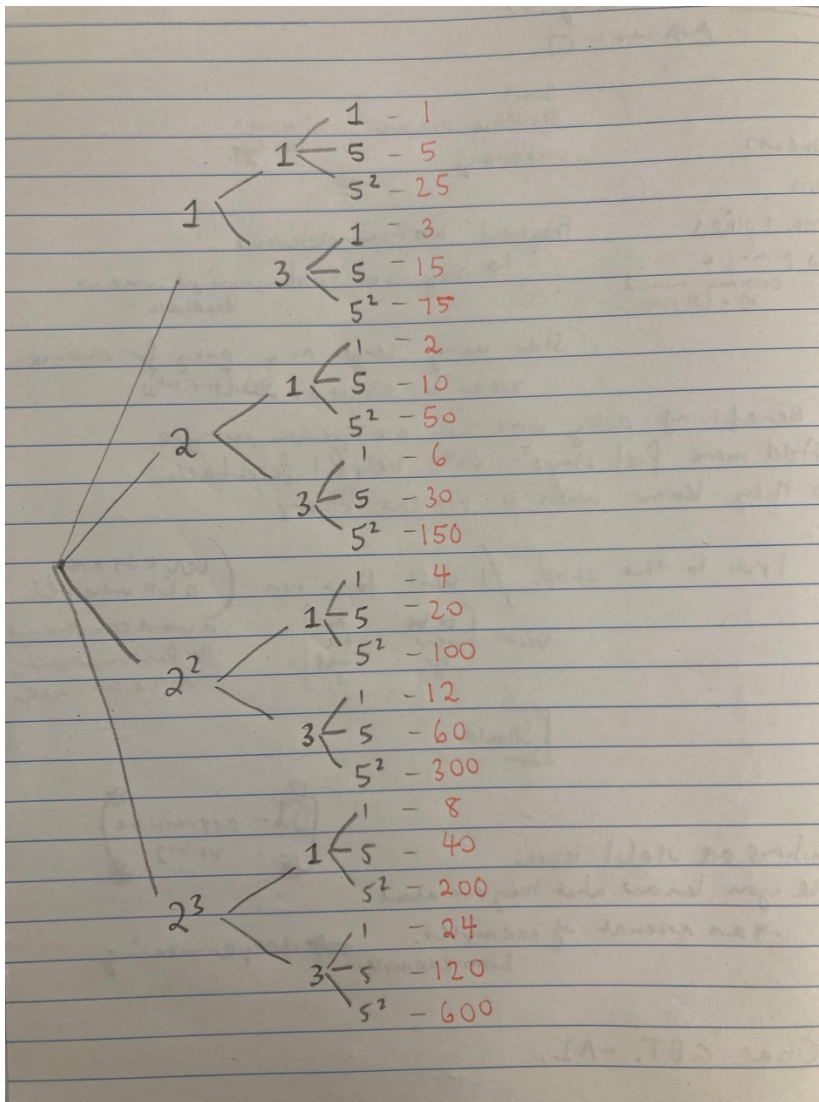
previous question. Again, we can use a tree diagram to systematically list the factors of numbers of the form  $2^n \times 3^m$ . Below we show a specific case for  $2^3 \times 3^1$ .



The general case will have  $n+1$  options for the powers of 2 ( $1, 2^1, 2^2, \dots, 2^n$ ) and  $m+1$  options for the powers of 3 ( $1, 3^1, 3^2, \dots, 3^m$ ). Since for every possibility of a power of 2 is paired with every possibility for a power of 3 we multiply to obtain the total number of options, and this is where the total of  $(n+1)(m+1)$  factors comes from.

What about  $2^n \times 3^m \times 5^r$ ? etc..

We again use a trusty tree diagram. We start with a "simple" case we can map out entirely, listing the factors of  $2^3 \times 3^1 \times 5^2 = 600$ , then consider the general case,  $2^n \times 3^m \times 5^r$ .



Again, in the general case we will have  $n+1$  options for the powers of 2 ( $1, 2^1, 2^2, \dots, 2^n$ ) and  $m+1$  options for the powers of 3 ( $1, 3^1, 3^2, \dots, 3^m$ ), and now we also have  $r+1$  options for the powers of 5 ( $1, 5^1, 5^2, \dots, 5^r$ ). The values again multiply to ensure that all possible combinations are accounted for (every possibility of a power of 2 goes with every possibility of a power of 3, goes with every possibility of a power of 5) and the number of factors of  $2^n \times 3^m \times 5^r$  as  $(n+1)(m+1)(r+1)$ .

This result holds regardless of how many prime numbers appear in the prime factorisation. While a complex formula, the most general case for the number of factors given a numbers prime factorisation say  $p_1^{q_1} p_2^{q_2} p_3^{q_3} \dots p_n^{q_n}$  can be expressed as follows:

$$(q_1+1)(q_2+1)(q_3+1) \dots (q_n+1)$$