

Problem 15

The following are the requirements for a stack of cans to be neat:

- The stack of cans has at least two rows
- The stack of cans is piled so that each new row has one less can than the row below it

What numbers can be neatly stacked?

This is our first question where we haven't been given a set of increasingly difficult questions to help us get started. I guess our first thought is, what would those questions be if they were here? Maybe it would be something like. Can you stack one can? Can you stack two cans? Can you stack three cans? Etc. Often starting with smaller cases, working systematically and looking for a pattern is a good way forward so let's do it.



















1: can't be stacked (can't have two rows or more with just one can!)

2: can't be stacked (can have two rows, but it isn't possible for the new row to have one less can than the one above it)

3: can be stacked (yay!)

If you haven't already, take some time to work through 4 cans up to 18 and see which ones work and which don't. Our answers are modelled on the next page.

Here's a summary of the results we found for 1-18 cans.

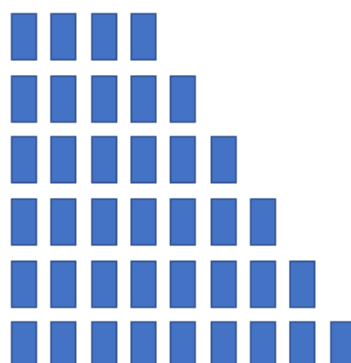
1	2	3	4	5	6
					
7	8	9	10	11	12
					
13	14	15	16	17	18
					

From these cases it appears that powers of 2 cannot be stacked, whereas all other numbers can be. We can't be certain that this pattern will always hold. But it's okay, we are promoted to provide a more rigorous answer in the Further Exploration...

Further Exploration: Why can some number be neatly stacked but others can't?

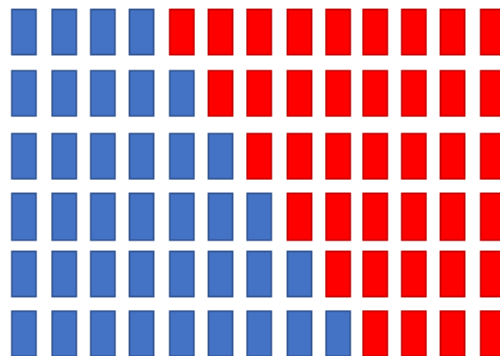
Let's start by constructing a formula that tells us how many cans are in a stack of base b and height h . Now, this is quite challenging and uses some quite complex algebra so let's go slowly and consider a specific case first.

Suppose we had a stack of cans that had a base containing 9 cans and a height of 6. This stack would have $9+8+7+6+5+4=39$ cans. Addition is a slow process, and if we work our way up to having situations like a base of 100 and a height of 73 we are going to have a lot of numbers to add. So, let's instead represent our stack in a different way, one where we can multiply to count the total number of cans.



We take this representation, make a copy (in red) and flip it over.

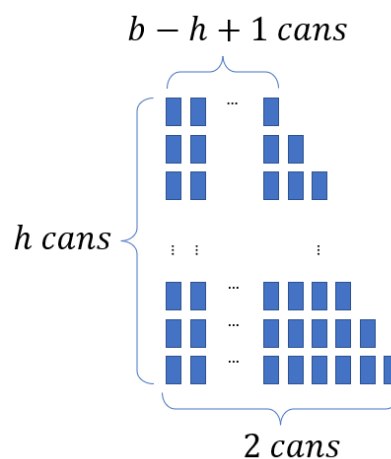
Note. This is very similar to what we did with our disc counting question in Problem 5!



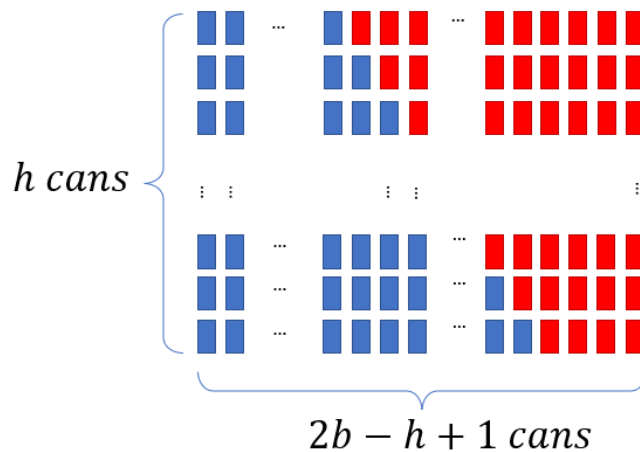
We have now made a rectangle where the base is $9+4=13$ and the height is 6. We can calculate the total number of cans by multiplying 13 by 6 and dividing by 2 (as we have two stacks, and we want to know how many cans are in one stack).

You might like to try this again for yourself by seeing if you can draw a picture and create a multiplication that counts the number of cans in a stack that has a base of 6 and a height of 4. Did you get $4 \times (6+3)/2 = 18$?

So how would this look in general? Well let's say we have a base of b , a height of h . Key to our calculation is knowing how many cans are in the top row of the stack. Well, if our base is b and there are h rows, the top row will have $b - (h - 1) = b - h + 1$ cans. Can you see why this is? Basically, for every row after the base we remove 1 can, so by the top we've removed $h-1$ cans. Removing $h-1$ cans is the same as removing h cans and giving one back, so we end up with $b-h+1$ in the top row.



We again, take this representation, make a copy (in red) and flip it over. We now have a formula for the general case.



The number of cans = $c = h \times (b+b-h+1) / 2 = h \times (2b-h+1) / 2$.

Now, we've been asked why some numbers can be stacked and why others can't. It turns out answering the "Why others can't" is easier so let's start there.

Currently we suspect that numbers of the form 2^n can't be stacked. That is:

$$2^n \neq h \times (2b-h+1) / 2$$

Well, 2^n is a number where every factor, other than 1, is even and so $h \times (2b-h+1) / 2$ must be a product of even numbers. For this to be true, h must be even. However, if h is even, what can be said about $2b-h+1$? Well, $2b$ is even (since $2 \times$ any number is an even number) and h is even so $2b-h$ is even. This means $2b-h+1$ must be odd – uh oh! It isn't possible for both h and $2b-h+1$ to both be even so it isn't possible for them to be equal to a power of 2. Ah ha, powers of 2 can definitely not be stacked!!

What about all other numbers, why can these be stacked? Well, every other number that isn't a power of 2 contains at least one odd factor. Let's write c for the number of cans and write $c=sk$ where s is the smallest odd factor of c and k is its factor pair. Now what follows took quite a lot of trial and error, though our biggest priority at any point was making sure that the height of the stack is less than or equal to the size of the base, that is $h \leq b$.

With this condition in mind, we separated the problem into two cases. Case 1: $s \leq k$ and Case 2: $s > k$.

Case 1:

If $s \leq k$ then we can choose the height of the stack to equal s (the smallest odd factor).

This gives:

$$\begin{aligned} c &= h \times (2b-h+1) / 2 \\ &= s \times (2b-s+1) / 2 \end{aligned}$$

Since $c = sk$, $k=(2b-s+1)/2$, rearranging this for b gives us a formula for the base of our stack $b=(2k+s-1)/2$. Now, this formula can only work if $2k+s-1$ is both even and greater than zero. The three components of our sum are $2k$, s and 1 . Since $2k$ is even, s is odd and 1 is odd the overall value must be even. Excellent! Also, since either s or k is greater than 2 the sum is positive. Fantastic!!

Now, the above information is hard to follow. Let's how it works in a specific case. Let's take a large number like 98. This number isn't a power of 2 so should be able to be stacked. Now let's check that

98 belongs in "Case 1." Well, 98's smallest odd factor is 7 and $98=7 \times 14$. Here we have $c=sk$ where $s=7$ and $k=14$ and $s < k$.

According to our formulas, we take $h=s=7$ and $b=(2 \times 14 + 7 - 1)/2 = 17$. Does a stack with a height of 7 and a base of 10 give a total of 98 cans? Well $c = h \times (2b - h + 1) / 2 = 7 \times (2 \times 17 - 7 + 1) / 2 = 98$. Yay!!

Case 2:

If $s > k$ then we choose the height of the stack to be $2k$.

This gives:

$$\begin{aligned} c &= h \times (2b - h + 1) / 2 \\ &= 2k \times (2b - 2k + 1) / 2 \\ &= k (2b - 2k + 1) \end{aligned}$$

Since $c = sk$, this time we have $s = 2b - 2k + 1$, rearranging this for b we now get $b = (s + 2k - 1)/2$. We need to ensure $s + 2k - 1$ is even and positive. The three components of our sum are s , $2k$ and 1 . Since s is odd, $2k$ is even and 1 is odd the overall value must be even. Excellent! Also, since either s or k is greater than 2 the sum is positive. Fantastic!!

The only other thing we need to be absolutely sure of is that $h \leq b$. Now $h = 2k$ and $b = (s + 2k - 1)/2$. Oh dear, this doesn't work!! There are cases like $20 = 5 \times 4$ where $s = 5$ and $k = 4$ and $s > k$. Using our formulas, we end up with a height of 8 and a base of 6, and such a stack doesn't exist!

To ensure that $b \geq h$ we must tighten up our conditions for our cases. Let's see what happens if we instead have Case 1: $s < 2k$ and Case 2: $s > 2k$. Notice how we don't need to say $s \leq 2k$ as s is odd and $2k$ is even.

In Case 1, is $s \leq (2k + s - 1)/2$? Well,

$$s < 2k$$

$$s \leq 2k - 1$$

$$2s \leq 2k + s - 1$$

$$s \leq (2k + s - 1)/2 \text{ YAY!!!!}$$

Let's consider Case 2. Here we have $s > 2k$, $h = 2k$ and $b = (s + 2k - 1)/2$. Is $h = 2k \leq (s + 2k - 1)/2$?

Well $s > 2k$ so...

$$s - 1 \geq 2k$$

$$2k \leq s - 1$$

$$4k \leq s + 2k - 1$$

$$2k \leq (s + 2k - 1)/2$$

Thank goodness!!

And so, we have it, all numbers not equal to a power of 2 can be stacked!!

Problem 16

- a. At a farm there are chooks and sheep. If there are 24 animals and 80 legs, how many sheep are there?

There are many ways to solve this problem, but we'll only talk about 3 of them.

One simple way is to guess and to improve on that guess until you get the right answer. For instance, suppose that there are 10 sheep. Then there would be 14 chooks. The number of legs here are $10 \times 4 + 14 \times 2 = 68$. This doesn't give us enough legs. The way to increase the number of legs is to have more sheep.

Suppose we guess 20 sheep. That will give us $20 \times 4 + 4 \times 2 = 88$. This is clearly too many sheep, so the next guess should be in between 10 and 20. Should the guess be nearer to 10 or 20?

Only a few more guesses should give you the answer.

Another way is to use algebra. Let c be the number of chooks and s be the number of sheep, then:

$$2c + 4s = 80 \quad \text{and} \quad c + s = 24.$$

Solving these equations will get you the answer.

The algebra seems to make the problem harder than it actually is...

Let's make a guess again.

Suppose there were 10 sheep, then there would be 14 chooks. This gives us $2 \times 14 + 4 \times 10 = 68$ legs. Now, *and this is the trick that we missed in the first solution*, each time we swap one chook for one sheep we get 2 extra legs. Since $80 - 68 = 12$ and $12/2 = 6$ we must need 2 more legs from 6 more sheep. This gives us 16 sheep and 8 chooks. Let's check this works:

$$2 \times 8 + 4 \times 16 = 80.$$

Isn't that neat? So, we don't need anything as advanced as algebra.

Further Explorations. How could you extend this problem?

This problem can be *extended* by making the numbers larger, or by adding a third animal. It can be *generalised* by creating a formula that works out the number of each animal for any given number of legs and animals.

(i) At a farm there are chooks and sheep. If there are 572 animals and 2028 legs, how many sheep are there?

Here we have just created a harder version of the same question by making the numbers larger. Any of the methods there will work, but we believe that the third method is the easiest so let's use that.

Say we had 572 chooks, this would give us 1144 legs. We needed 2028 legs and so we are $2028 - 1144 = 884$ legs off. We will get two extra legs for every chook that is replaced by a

sheep. This means we need to swap $884/2 = 442$ chooks for sheep. Now we have 130 chooks and 442 sheep. Does this work?

Well, $130 \times 2 + 442 \times 4 = 260 + 1768 = 2028$. Excellent!!

(ii) Two neighbouring farms each have chooks, sheep and spiders. Each chook is worth \$12, each sheep is worth \$130, and each spider is worth \$2.

- a) At Little Farm there are 19 animals, 114 legs and the total value of all animals is \$344. How many sheep are there?
- b) At Big Farm there are 920 animals, 6890 legs and the total value of all animals is \$6506. How many sheep are there?

This extends the problem by adding another layer of complexity. Note that if we add another animal, we need some extra information (here the cost of the animals) for there to be a single solution. [We talk about why this is at the very end of this problem.]

Let's look at little farm and see how our third method can be used to answer the question... Now, because the sheep are most expensive, it is easiest to consider how many sheep we could have.

0 sheep = \$0	<p>Guess: 19 chooks, giving 38 legs and costs \$228</p> <p>Not enough legs and not enough money. Since spiders are cheaper, replacing chooks with spiders won't help.</p>																				
1 sheep = \$130	<p>Guess: 18 chooks, giving 36 legs and costs \$216.</p> <p>We now have 19 animals, $2 \times 18 + 4 \times 1 = 40$ legs and $\\$216 + \\$130 = \\$346$</p> <p>For each chook we swap for a spider, we decrease the cost by \$10 and increase the number of legs by 6. This won't work here as we need an extra 74 legs but only have \$2 to play with.</p>																				
2 sheep = \$260	<p>Guess: 17 chooks, giving 34 legs and costs \$204.</p> <p>We now have 19 animals, $2 \times 17 + 4 \times 2 = 42$ legs and $\\$204 + \\$260 = \\$464$</p> <p>Again, for each chook we swap for a spider, we decrease the cost by \$10 and increase the number of legs by 6.</p> <p>We need an extra 72 legs, this requires 12 spiders. We also need to save $464 - 344 = \\$120$. This is our solution because having 12 spiders achieves this – great!</p> <p>Check:</p> <table><tr><td></td><td>Chooks</td><td>Sheep</td><td>Spiders</td><td>Total</td></tr><tr><td>Number</td><td>5</td><td>2</td><td>12</td><td>19</td></tr><tr><td>Legs</td><td>10</td><td>8</td><td>96</td><td>114</td></tr><tr><td>Cost</td><td>\$60</td><td>\$260</td><td>\$24</td><td>\$344</td></tr></table> <p>Success! The answer is that we must have 5 chooks, 2 sheep and 12 spiders.</p>		Chooks	Sheep	Spiders	Total	Number	5	2	12	19	Legs	10	8	96	114	Cost	\$60	\$260	\$24	\$344
	Chooks	Sheep	Spiders	Total																	
Number	5	2	12	19																	
Legs	10	8	96	114																	
Cost	\$60	\$260	\$24	\$344																	

3 sheep = \$390

Too expensive anyway!

As for big farm, we will use technology (and algebra) to help us.

Again, we will look at fixing the number of sheep (s), this time we can have anywhere from 0 to 50 sheep (as $51 \times \$130$ exceeds the value of the farm).

We will then set the number of chooks (c) to be the maximum amount. This is $c = 920 - s$.

From here we can work out the total number of legs (l) and the value (v).

$$l = 2c + 4s \text{ and } v = 12c + 130s$$

We'll then work out how far off we are in terms of legs and costs and see if replacing chooks with spiders works.

Extra legs required will be: $6890 - 2c + 4s$ and the missing value required is $6506 - 12c + 130s$.

If the extra legs divided by 6 is equal to the missing value divided by 10, we are all good! (To make this easy to find we find the difference between the number of spiders required to make the legs and value correct, when it's zero we know we are done).

Number of sheep	0	1	...	30	31	32	33	34
Number of chooks	920	919	...	890	889	888	887	886
Total number of legs	1840	1842	...	1900	1902	1904	1906	1908
Total value	11040	11158	...	14580	14698	14816	14934	15052
Extra legs required	5050	5048	...	4990	4988	4986	4984	4982
Excess value	4534	4652	...	8074	8192	8310	8428	8546
Spiders required to get extra legs	841.67	841.33	...	831.67	831.33	831	830.67	830.33
Spiders required to remove excess value	453.4	465.2	...	807.4	819.2	831	842.8	854.6
Difference in number of spiders required to make the number of legs and value correct	-388.3	-376.1	...	-24.27	-12.13	0	12.133	24.267

Why did we need another piece of information (the cost of the animals) to get a single solution.

Suppose we had a farm with spiders, sheep and chicken and we knew that there were 19 animals, 114 legs. Well let's start by focusing in on a specific number of spiders (as this reduces the problem right back down to very familiar territory with just sheep and chickens). We can have anywhere from 1 spider all the way up to 14 spiders (we can't have 15 spiders as this would give us 120 legs).

Number of spiders	Remaining animals and legs	Number of sheep and chicken
1	18 animals, 106 legs	Not enough legs (18 sheep, only 72 legs)
2	17 animals, 98 legs	Not enough legs (17 sheep, only 68 legs)
3	16 animals, 90 legs	Not enough legs (16 sheep, only 64 legs)
4	15 animals, 82 legs	Not enough legs (15 sheep, only 60 legs)
5	14 animals, 74 legs	Not enough legs (14 sheep, only 56 legs)
6	13 animals, 66 legs	Not enough legs (13 sheep, only 52 legs)
7	12 animals, 58 legs	Not enough legs (12 sheep, only 48 legs)
8	11 animals, 50 legs	Not enough legs (11 sheep, only 44 legs)
9	10 animals, 42 legs	Not enough legs (10 sheep, only 40 legs)
10	9 animals, 34 legs	8 sheep, 1 chook ($8 \times 4 + 1 \times 2 = 34$)

11	8 animals, 26 legs	5 sheep, 3 chooks (5x4+3x2=26)
12	7 animals, 18 legs	2 sheep, 5 chooks (2x4+5x2=18)
13	6 animals 10 legs	Too many legs (6 chooks, 12 legs)
14	5 animals, 2 legs	Too many legs (5 chooks, 10 legs)

We see from the table above that there are three different solutions for spiders, sheep and chooks with a total of 114 legs and 19 animals. It was specifying the cost that made only the 2 sheep, 5 chooks and 12 spider situation the only solution.

(iii) If a farm only has chooks and sheep, find an expression for the number of chooks for any given number legs and animals.

This final extension is actually a *generalisation*. A generalisation is where we work out what happens “in general.” A generalisation requires the use of algebra, or words.

This again requires knowledge of algebra. If we let c = the number of chooks, s = the number of sheep, l = the number of legs and a = the number of animals, then for any number of legs and animals we have the following solving the equations:

$$2c + 4s = l \quad \text{and} \quad c + s = a$$

To find an expression for the number of chooks we need to remove the number of sheep. We do this by writing the second equation as $s = a - c$ and replacing the s in the first equation with this.

$$\begin{aligned}
 2c + 4(a - c) &= l \\
 2c + 4a - 4c &= l \\
 -2c + 4a &= l \\
 4a - l &= 2c \\
 \frac{4a - l}{2} &= c
 \end{aligned}$$

Let's check that this formula works by returning to the original question. We had 24 animals and 80 legs and found that we needed 8 chooks. Well

$$\frac{4a - l}{2} = \frac{4 \times 24 - 80}{2} = \frac{96 - 80}{2} = \frac{16}{2} = 8$$

So good!

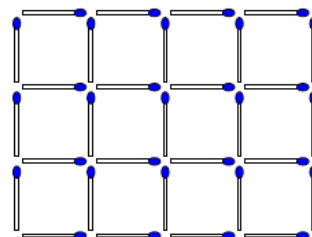
Now, if someone stopped you on the train and asked you what if $l = 288$ and $a = 39$, you'd be able to just give them the answer straight away - neat!

Problem 17

- a. How many matchsticks are required to make the 3 by 4 grid shown here?

One of the simplest ways here is to count all of the vertical matches along with the horizontal matches. In each of the five vertical columns there are 3 matches; in each of the four horizontal rows there are 4 matches. Combining these gives

$$3 \times 5 + 4 \times 4 = 31.$$



- b. How could you extend or generalise this problem?

This problem can be *extended* by making the numbers larger, or perhaps by adding a third dimension. It can be *generalised* by creating a formula that works out the number of sticks required for any size.

(i) Extend: How many matchsticks would be required to make a 13 by 33 grid?

Now we have 13 rows and 33 columns.

We can solve this problem the same way as part a. above. Focus on the vertical matchsticks and then look at the horizontal matchsticks.

What makes this problem harder is that we don't have a diagram to look at and so we can't just physically count the match sticks.

In problem a. there were 3 vertical matchsticks going down each time because there were 3 rows. Now we have 13 rows so we will have 13 matchsticks going down each time.

In problem a. there were 5 lots of each set of vertical matchsticks, this was one for each column plus an extra to complete the end. Here we will have 34 sets of vertical matchsticks. One for each of the 33 columns and one extra to complete the end.

This means we will have 34×13 vertical match sticks.

In problem a. there were 4 horizontal matchsticks going down each time because there were 4 columns. Now we have 33 columns so there will be 33 matchsticks going across each time.

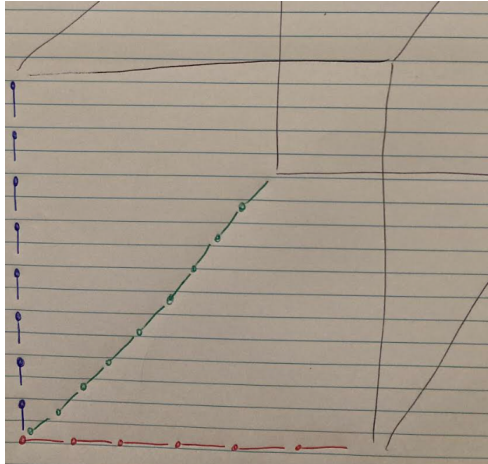
In problem a. there were 4 lots of horizontal matchsticks, this was one for each row plus one extra to complete the bottom. Here we will have 14 sets of vertical matchsticks. One for each of the 13 rows and one extra to complete the bottom.

This means we will have 14×33 horizontal match sticks.

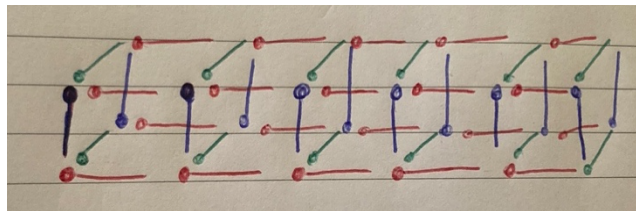
This gives a total of $13 \times 34 + 33 \times 14 = 904$ match sticks.

(ii) Extend: How many matchsticks would be required to 3D grid of length 6, height 8 and width 9?

Let's begin my making a little sketch.



To get more details we draw a 3D component of our shape, now this doesn't have the right dimensions, but just like how problem a. helped us with the first extension question, this picture can help us with our larger 3D grid.



Now we need to calculate how many horizontal (red), vertical (blue) and "into" (green) matchsticks there are in our 3D grid of length 6, height 8 and width 9.

Let's look at the horizontal matchsticks first. In the front layer we will have 6 horizontal matchsticks along every row, since the height is 8, we will have 9 rows of 6. With a width 9 we will have 10 layers (one for each row, plus one for the bottom). This gives us a total of $9 \times 6 \times 10$ horizontal matchsticks.

Let's now look at the vertical matchsticks. In the front layer we will have 8 vertical matchsticks along every column, since the length is 6, we will have 7 columns of 8 (one for each column, plus one for the end). With a width 9 we will have 10 layers. This gives us a total of $7 \times 8 \times 10$ vertical matchsticks.

We finally look at the "into" matchsticks. Let's consider the bottom layer first. The width is 9 so there are 9 matchsticks deep in each section. The cube is 6 long so we will have 7 groups of these (one for each length and one on the end). The height of the cube is 8 so we will require 9 layers (one for the bottom of each layer and the final top layer). This gives us $7 \times 9 \times 9$ "into" matchsticks.

The total number of matchsticks is $9 \times 6 \times 10 + 7 \times 8 \times 10 + 7 \times 9 \times 9 = 1667$.

(iii) Generalise: How many matchsticks would be required to make a m by n grid?

This can be done by the methods of part b to a general case. What did you get and why?

We got $m \times (n+1) + n \times (m+1)$

(iv) Extend: A square grid requires 480 matchsticks. What is the length of the square?

Suppose that the square has side t , then we have $t \times (t+1) + t \times (t+1) = 2t(t+1)$ matches. So
 $2t(t+1) = 480$ or $t^2 + t = 240$.

By trial and error you can get $t = 15$. (There is a negative value of t that we ignore because t has to be positive.)

Factorisation is another method, though finding the numbers 15/16 is not easy. Anyway, this approach produces $(t - 15)(t + 16) = 0$, so $t = 15$ and -16 . Clearly, we can reject the negative number.

Then there is always the quadratic formula. In this case $t = \frac{-1 \pm \sqrt{1 + 4 \times 240}}{2} = \frac{-1 \pm 31}{2} = 15$ or -16 .

Problem 18

- a. When $2/7$ is written as a decimal, what number is in the 30th decimal place?

This problem links to the power of 2 question, Problem 12. We certainly don't expect you to write out the first 30 decimal places of $2/7$. But maybe there is a pattern here that can save you doing all that work. If there is, how can we find it?

How about long division?

$$\begin{array}{r} 0.2857142857... \\ 7 \overline{) 2.0000000000...} \end{array}$$

By doing this little piece of long division the pattern that you see is the repetition of 2, 8, 5, 7, 1, 4. The pattern starts again after every 6 digits. The 30th digit ($30 = 6 \times 5$) is at the end of 5th cycle of 2, 8, 5, 7, 1, 4. So the 30th decimal place is a 4.

- b. How could you extend or generalise this problem?

This problem can be *extended* by making the numbers larger. It can be *generalised* by trying to describe what happens with different types of fractions and what types of decimals they produce.

- (i) Extend: When $2/7$ is written as a decimal, what number is in the 1000th decimal place?

From above, we know that the cycle of digits in the decimal representation of $2/7$ is 2, 8, 5, 7, 1, 4. So the question becomes, how many cycles do we need to get to the 1000th digit?

Well, 1000 divided by 6 = 166 remainder 4. This means we have 166 full cycles of 2, 8, 5, 7, 1, 4 and then we have four more digits which are 2, 8, 5, and 7. Making the 1000th digit of $2/7$ equal to 7.

- (ii) Extend: When $2/17$ is written as a decimal, what number is in the 1000th decimal place?

We need to work out what the digit cycle of $2/17$ is... let's use [Wolfram Alpha](#):

$$2 / 17 = 0.1176470588235294117647058823529411764705882352941176470588235294$$

You see that this pattern repeats every 16 digits where the cycle is "1176470588235294."

So now we again ask, how many cycles of 16 do we get in 1000?

1000 divided by 16 = 62 remainder 8. This means we have 62 full cycles and 8 more digits. The 8 digits are: 1, 1, 7, 6, 4, 7, 0, 5 and so the 1000th digit is 5.

- (iii) Generalise: What interesting thing(s) can be found out about the fractions $1/7$, $2/7$, $3/7$, $4/7$, $5/7$ and $6/7$.

- $1/7 = 0.1428571428571428571...$
- $2/7 = 0.2857142857142857142...$
- $3/7 = 0.4285714285714285714...$
- $4/7 = 0.5714285714285714285...$

- $5/7 = 0.7142857142857142857...$
- $6/7 = 0.8571428571428571428...$

Do you notice that the digits all complete the same cycle and just start at different points?

What do you think happens if we take even larger numerators?

(iv) Generalise: Do all fractions produce decimals that can be predicted in some way?

Let's start looking at some specific cases...

1/2	0.5	1/7	0.142857142857...	1/12	0.08333...
1/3	0.333....	1/8	0.125	1/13	0.076923076923...
1/4	0.25	1/9	0.111...	1/14	0.0714285714285...
1/5	0.2	1/10	0.1	1/15	0.0666...
1/6	0.1666...	1/11	0.090909...	1/16	0.0625

There seem to be two possible types. First the ones that stop, or, if you like, continue with zeros. Which fractions above produce decimals that stop?

We see $1/2$, $1/5$, $1/8$, $1/10$ and $1/16$.

Second the ones that get to a certain point and then keep repeating. Which fractions above produce decimals that go on forever in a pattern?

We see that it's everything except those listed above.

This raises the question of whether or not there are any other kinds of fractions. By a **fraction** we mean a number m/n , where m and n are positive integers. So does every fraction stop, like $1/10$ or repeat like $1/3$?

Experiment for a while with fractions to see if you can find a new kind that isn't like the two kinds we have discovered so far. Then consider what comes next.

Why do some fractions 'repeat'? That is, after some messing around, the same of numbers go round and round for ever. What forces this repetition? This must be because of a remainder being the same. Think about $2/7$ from our very first question. $2/7 = 0.28571428571428...$ After each of the 4s, in bold, there is the same remainder: 2. Think about $1/17 = 0.058823529411764705882352941176470588...$ Divide $1/17$ by hand and see when the remainder returns to 1 again. (We know that this is a pain, but there is a point for asking you to do this.) When that happens do you see the same set of twelve numbers repeating again, and then again, and ...

Why is this repeating necessary? Think. You are dividing by 17. What remainders can you have? How many remainders can you have? So do you see that a remainder must repeat every now and then. And when it repeats the same series of numbers repeat again.

Of course, some remainders are zero and the whole business comes to a stop (or goes on forever repeating zero.) So you can only get these two kinds of fraction.

But things are more powerful than this. It's actually true that the only numbers that have the sort of repetition we have been talking about are ... FRACTIONS. Why?

Suppose some number repeats like this 0.42142142142... Let x be this number. Then

$$x = 0.42142142142...$$

$$\text{Then } 100x = 421.42142142...$$

Subtracting x from $100x$ we get $99x = 421$.

Do you see why the right side is a whole number? OK, then. Now it must be true that

$$x = 421/99.$$

And that is a fraction!

Now experiment with any repeating group of numbers. Show that the number is a fraction.

Try $x = 0.0015615615615...$ Do you get a fraction again?

So every repeating decimal is a fraction. In mathematics we say that

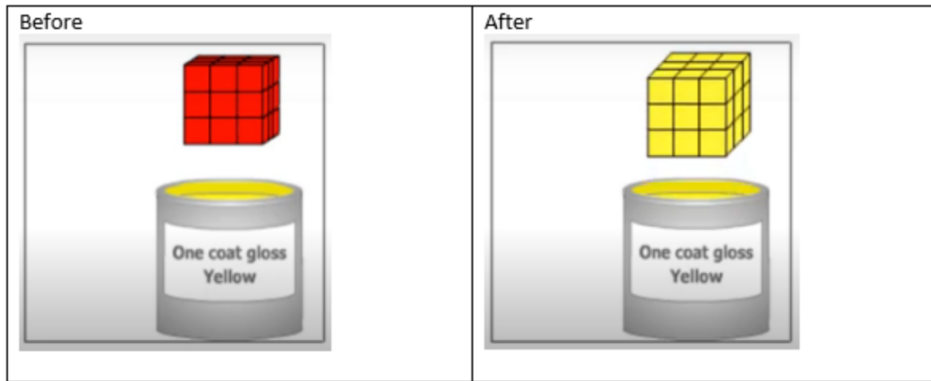
x is a fraction if and only it repeats (maybe with zero).

This generalises what we did in the question above.

And you might want to look at how long a repeating sequence is. What is the relation between the number of numbers in its repetitive part and the number itself?

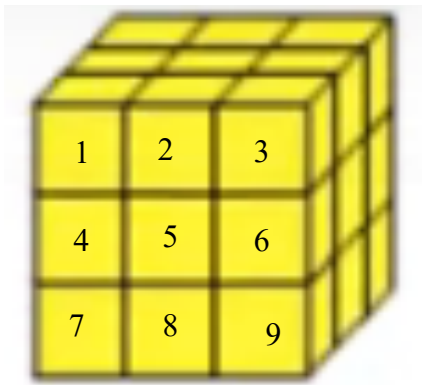
Problem 19

- a. A cube made of 27 red cubes is dipped in yellow paint so every face is covered.



Of the 27 cubes, how many will have zero yellow faces? How many will have one yellow face? How many will have two yellow faces? How many will have three yellow faces? How many will have four yellow faces?

Consider each of the nine labelled cubes shown below. How many yellow faces does each such cube have?



Well corner cubes, like 1, 3, 7 and 9, all have three yellow faces.
Edge cubes, like 2, 4, 6 and 8, all have two yellow faces.
Central cubes, like cube 5, have just one yellow face.

The question then becomes, how many corner, edge and central cubes does a 3x3x3 cube have?

We get 8 corner cubes, 12 edge cubes and 6 central cubes. This gives a total of 26, oh no – a 3x3x3 cube has 27 cubes, what have we missed? Oh yeah, the very centre of the cube (the cube behind 5). All is okay!! (and of course, we misread the question, we were asked how many cubes have zero yellow faces... revisiting a question after you think you have an answer is really worthwhile!)

- b. How could you extend or generalise this problem?

This problem can be *extended* by making the cubes larger. It can be *generalised* by talking about a cube of any dimension.

(i) Extend: What if the cube was $4 \times 4 \times 4$, $5 \times 5 \times 5$ or $6 \times 6 \times 6$? How many cubes will have no faces with yellow, one face with yellow, two faces with yellow and three faces with yellow?

Justify all the answers below and then move on to the $5 \times 5 \times 5$ and $6 \times 6 \times 6$ cubes.

In the $4 \times 4 \times 4$ case, the counts are:

- no faces with yellow: there is an inside unpainted $2 \times 2 \times 2$ cube. Giving 8 cubes with no faces painted.
- one face with yellow: each face of the larger cube has four of these. With 6 faces giving a total of $6 \times 4 = 24$.
- two faces with yellow: think about the edges, there are two edge cubes on each edge. There are 12 edges giving a total of $12 \times 2 = 24$.
- three faces with yellow: think about the corners, there are 8 corners (each made up of just the one cube).

How could you check that we have answered the question. Well first of all you could re-read the question and see if we've answered what was asked. At this point, we haven't answered it entirely, we've only just considered the $4 \times 4 \times 4$ cube. But focusing on the $4 \times 4 \times 4$ cube, do we have all cubes? Let's add up our answers and see if we have accounted for all 64 cubes. $8 + 24 + 24 + 8 = 64$. Awesome!

Here are the answers for $5 \times 5 \times 5$ and $6 \times 6 \times 6$. Can you use similar ideas to confirm these answers. Check carefully, we could have made a mistake...

$5 \times 5 \times 5$: no faces – 27, 1 face – 54, 2 faces – 36, 3 faces – 8.

$6 \times 6 \times 6$: no faces – 64, 1 face – 96, 2 faces – 48, 3 faces – 8.

(ii) Generalisation: What if the cube was $n \times n \times n$? How many cubes will have no faces with yellow, one face with yellow, two faces with yellow and three faces with yellow?

Having moved into extensions with bigger cubes, it would be good to be able to find all possible cubes. This is a generalisation.

We now look for answers to the questions that are essentially the answers for $n = 3, 4, 5$ and 6, that have now been done earlier here: How many cubes will have no faces with yellow, one face with yellow, two faces with yellow and three faces with yellow?

Check each of your answers by making sure that when you put $n = 4$, you get the answer that you have already found above.

Can $n = 1$ or 2? For completeness check these values too.

Let's go through a possible solution:

- no faces with yellow: there is an inside unpainted $(n-2) \times (n-2) \times (n-2)$ cube. Giving $(n-2)^3$ cubes with no faces painted. Why are the dimensions of the unpainted cube all $n-2$? Well, you need to remove the whole outer layer of yellow painted cubes. This requires all the bottom and top to be removed, and all the left and right, and all the front and back. This reduces the dimension of the cube by 2 in each direction.
- one face with yellow: each face of the larger cube has $(n-2)^2$ of these. With 6 faces giving a total of $6 \times (n-2)^2$. The reasoning for the $n-2$ is much the same as above. To expose the one face painted parts, we must remove the edges and corners.

- two faces with yellow: think about the edges, there are $(n-2)$ cubes on each edge. Each edge is the dimension of the cube with the corners removed (2 corners per edge gives $n-2$ per edge). There are 12 edges giving a total of $12 \times (n-2)$.
- three faces with yellow: think about the corners. No matter how big our cube is there are always 8 corners (each made up of just the one cube).

In summary: $n \times n \times n$: no faces – $(n-2)^3$, 1 face – $6 \times (n-2)^2$, 2 faces – $12 \times (n-2)$, 3 faces – 8.

There are a couple of ways we can check if this answer works. Firstly, we could substitute $n = 3$ in and see if we get the same answers as our original problem. Let's do that.

- No faces: $(n-2)^3 = (3-2)^3 = 1$ ✓
- One face: $6 \times (n-2)^2 = 6 \times (3-2)^2 = 6$ ✓
- Two faces: $12 \times (n-2) = 12 \times (3-2) = 12$ ✓
- Three faces: 8 ✓

The other way we can check is by adding up the values and seeing if we get n^3 . Now this requires a bit of algebra, the process of expanding more specifically. You might like to do a little research on expanding then have a read to see how we have used expanding to get our answer. We've used colours to show where each part comes from.

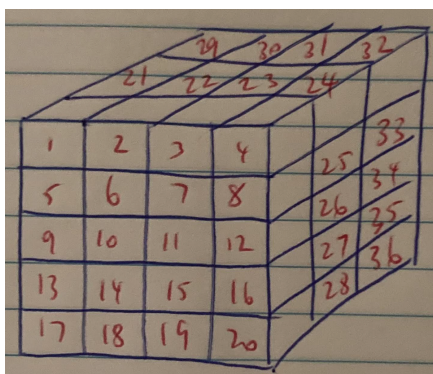
$$\begin{aligned}
 & (n-2)^3 + 6 \times (n-2)^2 + 12 \times (n-2) + 8 \\
 &= n^3 - 6n^2 + 12n - 8 + 6 \times (n^2 - 4n + 4) + 12n - 24 + 8 \\
 &= n^3 - 6n^2 + 12n - 8 + 6n^2 - 24n + 24 + 12n - 24 + 8 \\
 &= n^3
 \end{aligned}$$

Pretty cool seeing how everything cancels (that is $-6n^2$ with the $+6n^2$ and $12n$ with $-24n$ and $12n$) and how just the n^3 is left behind!

(iii) Generalisation: What if the shape was a rectangular prism with sides p , q and r ? How many cubes will have no faces with yellow, one face with yellow, two faces with yellow and three faces with yellow?

This is a further generalization because it will cover all prisms. (Are there any limits on p , q and r ?)

If you are having trouble with this one why not start by considering a specific case. It seems that interesting things happen once the dimensions are three or bigger (as otherwise we miss out on cubes which only have one side painted), so let's start with a prism that is $5 \times 4 \times 3$. Let's draw a picture.



Let's consider the four categories: corner, edge, face and inner.

Corner cubes, like 1, 4, 17, 20, 29, 32 and 36, all have three yellow faces. As with any cube, any prism will always have 8 corner cubes.

Edge cubes, like 2, 3, 5, 9, 13, 8, 12, 16, 30, 31, 33, 34, 35, all have two yellow faces. Counting all the edge cubes is complex as there are three different lengths of edges. Excluding the corner cubes, the lengths of the edges are 1 (those that run through, like 21, 24 and 28), 2 (those that run along the top or bottom, like 2&3, 18&19 and 30&31) and 3 (those that run along the sides, like 5&9&13, 8&12&16 and 33&34&35). Remember that there are 12 edges on a prism so there are four of each of the 3 types. This gives us $4 \times (1 + 2 + 3) = 24$ edge cubes, each with 2 faces painted.

Face cubes, like cube 6, 7, 10, 11, 14, 15, 22, 23, 25, 26, 27, all have just one yellow face. Counting the number of face cubes is again tricky because there are three different face dimensions. The front face has dimension 2×3 (6, 7, 10, 11, 14, 15), the top face has dimension 1×2 (22, 23) and the right face has dimension 1×3 (25, 26, 27). There are 6 faces on a prism and so there are two of each of the three types. This gives us $2 \times (2 \times 3 + 1 \times 2 + 1 \times 3) = 22$ face cubes, each with one face painted.

Inner cubes are considered last. We can't refer to any numbers as we can't see them! However, we know that there is a $3 \times 2 \times 1$ prism directly behind cubes 6, 7, 10, 11, 14 and 15. This gives us 6 cubes with no faces painted.

Let's check we have done this right by adding up all our answers. Since the dimension of the prism is $5 \times 4 \times 3$ we should have 60 cubes altogether. $8 + 24 + 22 + 6 = 60$. Fantastic!!

Can you see how these ideas might be generalised? We'll leave this solution to you. If you are keen to hear our thoughts email us. Wendy.taylor2@education.vic.gov.au.

Problem 20

A fraction with a numerator of 1 is called a unit fraction. Here are sums of different unit fractions which make $1/6$.

$$1/7 + 1/42 = 1/6 \quad 1/8 + 1/24 = 1/6 \quad 1/9 + 1/18 = 1/6 \quad 1/10 + 1/15 = 1/6$$

- a. How many ways can you write $1/8$ as the sum of two different unit fractions?

Let's look at the first answer from above: $1/7 + 1/42 = 1/6$. We see that the first denominator is one bigger than the denominator of the answer and that the denominator of the second fraction is the product of the other two denominators. We wonder if something similar happens for $1/8$. Does $1/9 + 1/72 = 1/8$? Check it out!

This is a great start, but we wonder if more will work. Do you notice anything else about the sums above? Yeah, the denominator in the first fraction of the sum is increasing by 1 each time. We know $1/9 + 1/72 = 1/8$, can we also write $1/10 + ? = 1/8$ and $1/11 + ? = 1/8$ and $1/12 + ? = 1/8$ and $1/13 + ? = 1/8$ and... We wonder how many of these will work and also wonder when should we stop??

To work out the ? value in each equation we need to work out how much needs to be added to make $1/8$, the easiest way to do this is to use subtraction! Let's go..

- $1/8 - 1/10 = 1/40$
- $1/8 - 1/11 = 3/88$
- $1/8 - 1/12 = 1/24$
- $1/8 - 1/13 = 5/104$
- $1/8 - 1/14 = 3/56$
- $1/8 - 1/15 = 7/120$
- $1/8 - 1/16 = 1/16$

Wow, not nearly as many work as before. Oh well at least we have a few overall:

$$1/9 + 1/72 = 1/8 \quad 1/8 - 1/10 = 1/40 \quad 1/8 - 1/12 = 1/24$$

Note that we don't include $1/16 + 1/16 = 1/8$ because we have been asked for two different unit fractions.

- b. How could you extend or generalise this problem?

This problem could be *extended* by making the denominator of the fraction larger. However, since we already have two cases explored we think it makes sense to go straight into the generalisation, looking at how we can express $1/n$ as the sum of two different unit fractions.

(i) Generalisation: Is it true that $1/(n+1) + 1/[n(n+1)] = 1/n$?

Let's do the algebra:

$$\frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{n}{n(n+1)} + \frac{1}{n(n+1)} = \frac{n+1}{n(n+1)} = \frac{1}{n}$$

Fantastic result!! This means that we can write any unit fraction as the sum of at least one pair of different unit fractions.

(ii) Generalisation: Is it always possible to express a unit fraction as the sum of two different unit fractions in more than one way?

Recall from our work in part a. that it was easiest to work with subtractions than additions, that is if we have a unit fraction $1/n$ and we wish to determine whether we can write it as a sum involving $1/x$, that is $1/n = 1/x + ?$ we can calculate $1/n - 1/x$. Now in general

$$\frac{1}{n} - \frac{1}{x} = \frac{x}{nx} - \frac{n}{nx} = \frac{x-n}{nx}$$

This means that as long as we can find a value of x , other than one, such that $x-n$ is a factor of nx then we will obtain another unit fraction sum that works. Question is, how many fractions does this work for?

It will be helpful to start with specific cases and see if we can find a pattern.

Let's start with documenting our results so far, then consider some other values of n . Recall that we only need to consider up to $x = 2n-1$ because at this point we have $(x-n)/nx = 1/2n$ and the fractions in the sum are the same.

$n = 6$

x	7	8	9	10	11
x-n	1	2	3	4	5
nx	42	48	54	60	66
Is x-n a factor of nx?	Yes	Yes	Yes	Yes	No

$n = 8$

x	9	10	11	12	13	14	15
x-n	1	2	3	4	5	6	7
nx	72	80	88	96	104	112	120
Is x-n a factor of nx?	Yes	Yes	No	Yes	No	No	No

$n = 9$

x	10	11	12	13	14	15	16	17
x-n	1	2	3	4	5	6	7	8
nx	90	99	108	117	126	135	144	153

Is $x-n$ a factor of nx ?	Yes	No	Yes	No	No	No	No	No
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$n = 10$

x	11	12	13	14	15	16	17	18	19
$x-n$	1	2	3	4	5	6	7	8	9
nx	110	120	130	140	150	160	170	180	190
Is $x-n$ a factor of nx ?	Yes	Yes	No	Yes	Yes	No	No	No	No

$n = 11$

x	12	13	14	15	16	17	18	19	20	21
$x-n$	1	2	3	4	5	6	7	8	9	10
nx	132	143	154	165	176	187	198	209	220	231
Is $x-n$ a factor of nx ?	Yes	No	No	No	No	No	No	No	No	No

Okay, so we've found our first example where only one possible sum exists and here n is a prime number. We wonder if we can say that all unit fractions where the denominator is a composite numbers can be written in more than one way whereas all unit fractions where the denominator is a prime number cannot??

Creating similar tables (using excel) for $n = 3, 5, 7$, and 13 gives further evidence that prime denominators don't work, so let's go back to the algebra. Recall that to create a unit fraction we require $x-n$ to be a factor of nx . Now we know that $x-n$ won't be a factor of n as n is a prime number, so we only need to convince ourselves that $x-n$ is not a factor of x for $x = n+2, \dots, 2n-1$. Well as can be seen from the table above, if x is even $x-n$ will be odd so clearly two will never be a common factor. What about when x is a multiple of 3? Say $x=3k$. Well $n-x=n-3k$, since n is not a multiple of 3 (because n is prime), $n-3k$ can't be either. This same logic works when x is a multiple of 4, or 5, or any number. Consequently, x and $n-x$ cannot share any factors and so if n is prime then there is no other solution other than when $x = n+1$.

As for the composite numbers, suppose $n=ab$ where $a < b$ and $a > 1$ and $a < n$. As can be seen by each of the tables above, the values of $x-n$ always vary from 1 to $2n-1$ and since $a > 1$ and $a < n$ then a will be some value in this set not equal to 1. We know that $x-n=1$ will always give us a solution (see generalisation (i) above), and now we know that choosing $x-n=a$ will give another solution (as $x-n = a$ and a is a factor of n and so will also be a factor of nx). So we always have at least two ways of writing the sum of n , the denominator of the unit fraction, is a composite number.

In summary, we can only express $1/n$ as the sum of two different unit fractions in more than one way if n is a composite number.

