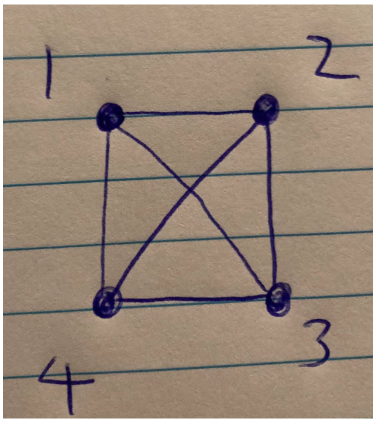
Problem 21

If there are 4 people in a room and everyone was to fist bump every other person, 6 fist bumps would occur. How many fist bumps would occur if there were 100 people in the room?
Note: A fist bump is a "young person's handshake" ☺

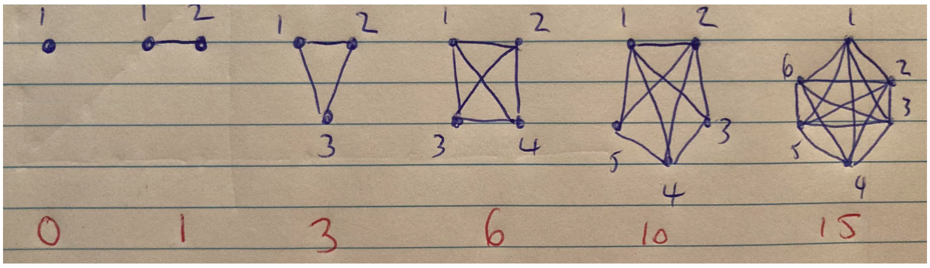
Let's start by making sure we understand why 6 fist bumps will occur between 4 people. Here's a helpful diagram.



Now, with other problems we've worked on, it was helpful to start with smaller cases, work systematically and look for patterns. So let's work out how many fist bumps occur if there was 1 person, 2 people, 3 people etc. You might like to draw diagrams like the one above too.

Have a go yourself then look at the next page to see if you get what we got.

Here's what we got:



Number of people	1	2	3	4	5	6
Number of fist bumps	0	1	3	6	10	15

Well, this is a great start. While drawing the diagrams is getting complex, it is great to see a lovely pattern forming in our table. What pattern do you see? How would you explain why it is happening?

We have two different explanations.

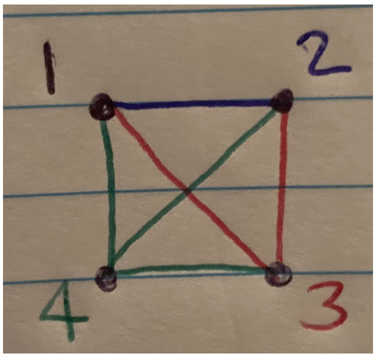
The first looks at our diagrams. Let’s zoom in on the diagram for four people. How many lines can you see coming out of the first dot? We see three. How many lines can you see coming out of the second dot? Again three? There are three lines coming out of every dot so we have 4 x 3 = 12 lines. Uh-oh, the original Problem (and the table above!) said that there were ‘6’ fist bumps between four people. Where did it make a mistake? Or did we make the error?

Why do you have to divide 12 by 2 to get the right answer? Ah, because using this method we count every fist bump twice, and we should have only counted each once. Diving the total number of lines coming out of every dot and dividing it by 2 will work a treat. This is great as now we have a way of working out how many fist bumps occur between 100 people. Have a go at working it out and then read on.

Do you get 4950, or something completely different. Well, we calculate the number of fist bumps as 100 x 99/2 = 4950. We get this because there are 99 lines coming out of each of the 100 dots, and we divide by 2 because 99 x 100 counts all lines twice.

But surely now we can give the general answer for n people and the number of fist bumps. This ought to be n(n -1)/2.

Here's another way to count the lines without needing to count them twice. Let's look at this coloured diagram.



When it's just person 1 they don't have anyone to fist bump. When person 2 comes along they can fist bump person 1 (one blue line). When person 3 comes along they can fist bump person 1 and person 2 (the two red lines). Lastly, when person 4 comes along they can fist bump person 1, 2 and 3 (the three green lines).

Now all of the lines have been coloured and the number of lines is 1 + 2 + 3 = 6. No duplications here.

We can use a similar approach for any number of people. If there were 100 people, we'd have 1 + 2 + 3 + … + 99

We see that this method gives us a sum of consecutive numbers, just like we got in Problem 5. For this problem we saw that the sum of the number 1 + 2 + 3 + … + n = n(n+1)/2. This is a little different from the formula we got above which was n(n-1)/2. Can you work out the reason for this?

The difference comes about because even though we have 100 people, we only need to sum from 1 to 99 because each person fist bumps the all the people from the first to the person just before them.

So far we have solved the problem (how many fist bumps occur if there are 100 people: 4950), and also **generalised** the problem (how many fist bumps occur if there are n people: n(n-1)/2). I wonder if we can **extend** this problem….

What if every 3 people in the group of 4 people hugged. Call this a **3-hug**. How many 3-hugs were there? What if there were six people in the overall group? What about 100 people? What about n people 3-hugging?

4 people: abc, abd, acd, bcd.

6 people: abc, abd, abe, abf, acd, ace, acf, ade, adf, aef, … You'll have to be really systematic to get 20.

100 people: 161,700 – I don't see you using a systematic method.

n people: $n(n-1)(n-2)/(3 \times 2 \times 1)$.

So how do you get the generalisation? First of all, you can choose any n people to get started. Then you can choose another n – 1 people then another n – 2 people. That looks like

$$n(n-1)(n-2)$$

But just in the case of fist bumps, there are repetitions. This is because of the order you choose the people. For example: abc will be chosen as abc, acb, bac, bca, cab and cba. So, every count/choice we had so far needs to be divided by 6 = 3 x 2 x 1. This gives the answer above.

But why stop at 3-hugs. How many m-hugs can be made with a group of n people (where $n \geq m$)? Enjoy!!

Problem 22

There are spiders, pigs and chooks on a farm. There are 32 legs, how many of each animal could there be?

This is another extension of Problem 16. Though here we are given three animals and only two pieces of information which allows the problem to have even more solutions!!

Let's summarise the results in a table:

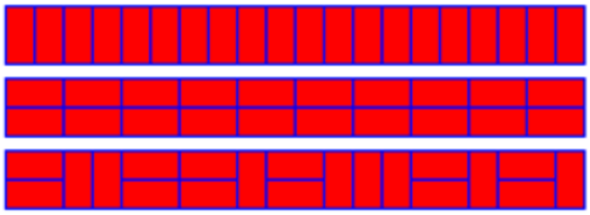
Number of spiders	Remaining legs	Possible number of sheep and chooks
0	32	8 sheep + 0 chooks, 7 sheep + 2 chooks, … , 1 sheep + 14 chooks, 0 sheep + 16 chooks (9 different solutions)
1	24	6 sheep + 0 chooks, 5 sheep + 2 chooks, …, 1 sheep + 10 chooks, 0 sheep + 12 chooks (7 different solutions)
2	16	4 sheep + 0 chooks, 3 sheep + 2 chooks, … , 1 sheep + 6 chooks, 0 sheep + 8 chooks (5 different solutions)
3	8	2 sheep + 0 chooks, 1 sheep + 2 chooks, 0 sheep + 4 chooks (3 different solutions)
4	0	0 sheep + 0 chooks (1 different solution)

There are a total of 25 different solutions.

I wonder how many solutions there would be if there were 1000 legs??? What if there were 4n legs? There's actually a really nice pattern in these numbers. I wonder if you can find it…

Problem 23

We have to build a footpath, 60cm by 6m, out of bricks that are 30cm by 60cm. The bricks can lie vertically and horizontally, but in no other direct. Three possible brick arrangements are shown below. How many different ways are there to build the walkway?



To solve this problem, we recommend completing the table below.

Length of path (cm)	30	60	90	120	150	180	210	240	270
Number of tilings	1	2	3						

Complete the table as far as you need to go until you have either completed the whole table or you have a pretty good idea of what is going on here. If not consult Leo.

Let D(e) be the number of tilings for a footpath that is 30e cm long. From the table, D(1) = 1 and D(2) = 2. Then D(3) can be thought of as the number of footpaths starting from the 30cm mark plus the number starting from the 60cm mark. Isn't that D(2) + D(1)? Can we get D(4) = D(3) + D(2)?

This is beginning to look like a Fibonacci problem. D(1) = F(2), D(2) = F(3) and D(e) = F(e + 1). Does D(e) = D(e – 1) + D(e – 2).

Now invent two more problems that are involved with Fibonacci numbers. What is the general term of your problems.

It turns out that 1089 x 9 = 9801. In other words, 9 flips round the number 1089. Find all of the flips caused by 9 that have three, four, five and so on digits. Is it true that for eighth digits the number of flips is three: 1099999989, 1098910989, 1098001098?

How many 10-digit numbers can be flipped by 9?

Generalise; extend.

Problem 24

To solve the puzzle of the Tower of Hanoi, you are required to move all disks to the third stick, whilst never placing a larger disk onto a smaller disk.

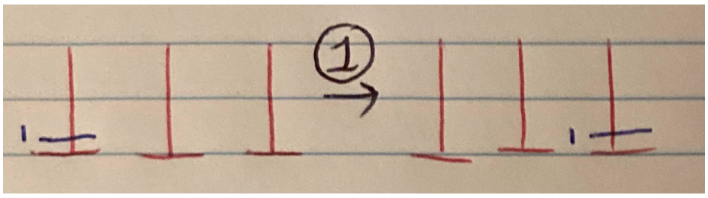
a. What is the smallest number of moves in which this puzzle can be completed?

b. What if there were 10 disks instead of 3? What is the smallest number of moves in which this puzzle can be completed?

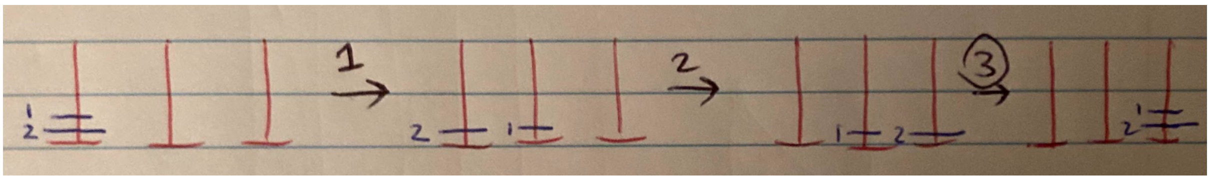
The first move is crucial. Where might the smallest disc move to? Of those moves which is best? What might happen next for the second smallest disc? Where would you move some middling sized disc? Why?

Below we show the moves for the first three cases.

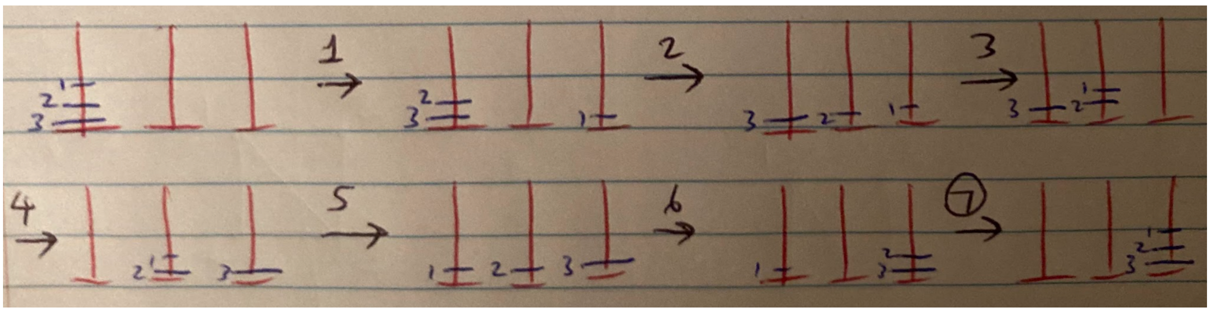
One disc:



Two discs:



Three discs:



From here try to complete the 4-disc situation. Would you move the smallest disc to second stick or the third? What about the next disc? Is it easy from here? Now try 6, 7, 8, 9. For 9 discs the number of moves is in the low thousands. This might be too large to be accurate, but try it anyway. There may be some way that you can see how to move different blocks of discs. These might reduce the amount of work required.

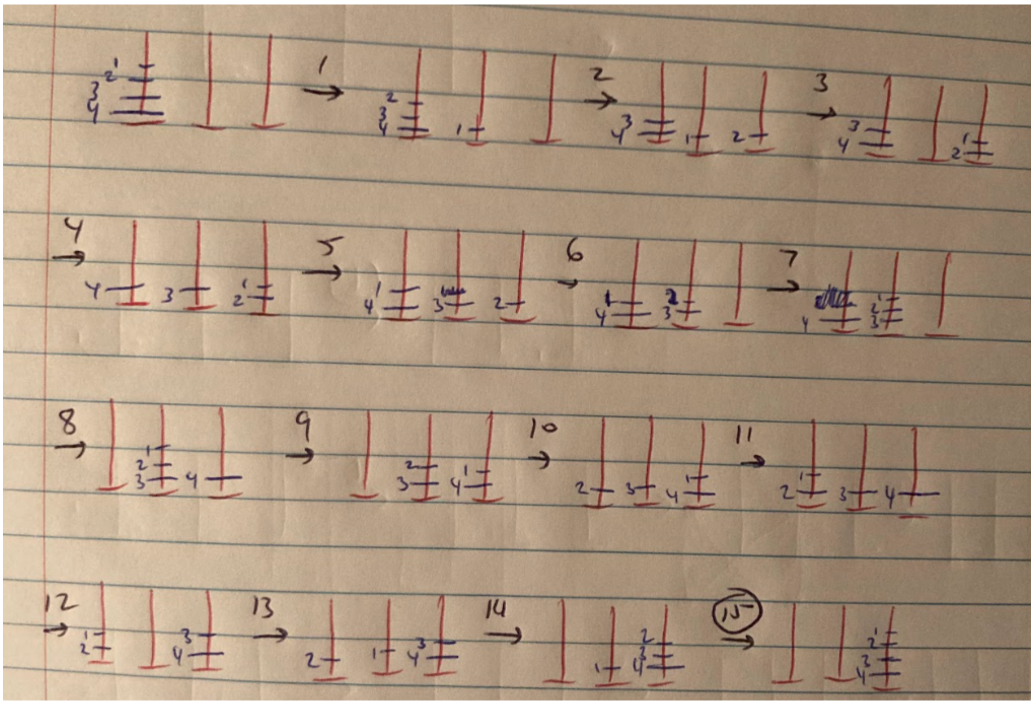
It would be useful to draw up a table and see what values you can complete. What pattern can you see?

Table: The number of moves needed for a given number of discs.

Number of discs	1	2	3	4	5	6	7	8	9
Number of moves	1	3	7						

What can you learn from the table? What is the pattern here?

Let’s draw out the case for four discs and look for clues.



Do you notice how after 1 step the 1-disk case is made, after 3 steps the 2-disk case is made, after 7 steps the 3-disk case is made, and of course after 15 steps the 4-disk case is made.

The fact that we have to build previous towers when building a new one gives a very big clue as to how we can calculate the total number of moves.

With this knowledge, it is possible to see how we can build the 5-disk tower just by being able to build the 4-disk tower. Ignoring the bottom, largest disk for now, we know that it takes 15 moves to build the 4-disk tower on the third stick. The positions of the sticks doesn’t affect the number of moves so we could also build the 4-disk tower on the second stick in 15 moves. From here we can move the largest disk to the third stick in one move and then use another 15 moves to move the tower from the second stick to the third, and doing so complete our puzzle. We have now shown that we can build the 5-disk tower in 31 moves.

We can use this argument for any number of disks. Suppose that we can move all but the bottom, largest disk in A moves then the total number of moves to move all of the discs would be

$$A + 1 + A = 2A + 1.$$

Here the first A gets the block part onto the second stick; the 1 move gets the largest disc to third stick; and the second A gets the block onto the third stick.

If we knew A we could get probably the block step by step both times. So let's do this with a block of one disc. To move the one smallest disc to Tower 2 takes 1 move. Then the other block goes to Tower 3. Then the smallest block goes to Tower 3 in 1 move. So

2 discs can get to the third stick in $2 \times 1 + 1 = 2 + 1$ moves.

To move three discs, let the block consist of the two smallest discs. Then we know that A here is 3 – we just showed that. We can move the block here across the Towers in 3 moves. As a result

3 discs can get to the third stick in $2 \times (2 \times 1 + 1) + 1 = 2 \times 2 + + 2 \times 1 + 1 = 2^2 + 2 + 1$ moves.

For 4 discs we get

$$2^3 + 2^2 + 2 + 1.$$

It looks as if for d discs we might get the result

$2^{d-1} + 2^{d-2} + \ldots + 2^2 + 1$, which can be simplified to $2^d - 1$. (How?)

So let's turn this into a formal proof. What we want to do in this proof is to know first that for k discs we can get the third stick filled in $2^k – 1$ moves.

This is certainly true for k = 1, because $2^1 – 1 = 1$.

Now we've suggested that for k discs we get $2^k – 1$ moves.

So what happens with k + 1 discs. Well, by the method above $A = 2^k – 1$, so for k + 1 discs we get

$$2A + 1 = 2(2^k – 1) + 1 = 2^{k+1} – 2 + 1 = 2^{k+1} –1.$$

This type of proof is called a **Proof Mathematical by Induction**. Let's see how this works.

First, we showed the formulas was right for k = 1.

Second, the proof above showed that the formula worked for k + 1 = 1 + 1 = 2.

Third, now we can put k = 2 because we have just proved that that works. And this shows that the formula is true of k = 3!

If you keep on going round the argument, next we can show everything is cool for k = 4, then 5, then 6, then …. Actually then forever. So for all d the number of moves is $2^d –1$.

At this point it is much easier to find the number of moves needed to solve the 10 disc problem. The answer is $2^{10} – 1 = 1023$. Did you have the patience and systematic skill to get this number with actual disc moving?

On the side: This Proof by Mathematical Induction can show that in the big dot diagram in Question 21 with n big dots has n(n – 1)/2 lines. It goes this way.

First if there is one dot then there are zero (1 x 0/2) lines.

Assume for k big dots, that the number of lines is $k(k - 1)/2$.

Then for k + 1 big dots, we have $k(k - 1)/2$ lines in a closed section with k big dots. Then the (k + 1)th dot has a line from itself to every other big dot. So in the k + 1 situation there are

$$k(k - 1)/2 + k = (k/2)[k - 1 + 2\} = k(k + 1)/2 = (k + 1)(k + 1 - 1)/2 \text{ lines.}$$

So for n big dots we get $n(n - 1)/2$.

Check this proof out by looking at k = 1, k = 2, k = 3 and so on forever.

Extension. In the Tower of Hanoi problem, let's change one thing. Let's add a fourth stick. How many moves are needed to move the n discs from stick 1 to stick 4?

There is a problem here. No one seems to know how to find a formula for this. Can you be the first?